2017

An Engage or Retreat differential game with Mobile Agents

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An Engage or Retreat Differential Game with Mobile Agents

A Thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Electrical Engineering

by

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B.E., Osmania University, 2013

2017
Wright State University

Wright State University
GRADUATE SCHOOL

July 24, 2017

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Swathi Chandrasekar ENTITLED An Engage or Retreat Differential Game with Mobile Agents BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Electrical Engineering.

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ABSTRACT

Chandrasekar, Swathi. M.S.E.E, Department of Electrical EngineeringWright State University, 2017. *An Engage or Retreat Differential Game with Mobile Agents.*

The thesis is aimed at developing optimal defensive strategies that dissuade an attacker from engaging a defender while simultaneously persuading the attacker to retreat. A two-player *Engage or Retreat* differential game is developed in which one player represents a mobile attacker and the other player represents a mobile defender. Both players are modeled as massless particles moving with constant velocity. The choice to terminate the game in engagement or retreat lies with the attacker. The defender indirectly influences the choice of the attacker by manipulating the latter's utility function. In other words, the defender co-operates with the attacker so that retreat appears to be the best option available. The solution to the differential game is obtained by solving two related optimization problems namely the *Game Of Engagement* and *Optimal Constrained Retreat*. In the Game of Engagement, the attacker terminates the game by capturing the defender.In the Optimal Constrained Retreat, a value function constraint is imposed which deters the attacker's retreat trajectory from entering into a region where it may lead to engagement. Such regions where constrained retreat occurs are known as *escort regions*. The solutions to these two problems are used to construct the global equilibrium solutions to the *Engage or Retreat* differential game.The global equilibrium solution divides the admissible state space into two regions that contain qualitatively different equilibrium control strategies. Numerical solutions are included to support the theory presented.

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Acknowledgment

I would like to take this opportunity to profusely thank my parents for being a pillar of support and strength to me.

I thank my advisor Dr.Fuchs for being patient and standing through thick and thin with me. I also take this opportunity to thank my committee members. I appreciate the support I received from my coresearch workers Zack, Pavlos, David, Bikash, Sam and Eric who made research seem fun. I appreciate and value the support of my friends and roommates-Achyuth,Vibha,Rachitha, Pragathi and Suneesh. I also thank the almighty for being my source of strength and guiding light.

Dedicated to

My Mother, My Father, Dr.Fuchs and the Almighty

Introduction

During combat situations, many a time, a situation arises where the protection of high value assets poses a challenge. If the assets are mobile, the complexity is further increased. The assets must produce a credible threat for the attacking agents such that the attack never occurs. This means that the defending assets must offset any reward or benefit the attacking agents would gain. If such a scenario could be created, retreating would be a more suitable agents for the attacking agents.

The attacker must engage when the reward for attacking is greater than the cost of attacking the defender. In order to protect itself, the defender could increase the cost of the attacker so that it supersedes the reward the attacker would achieve if it were to attack. This strategy of indirectly manipulating the cost of the attacker provides the defender an indirect form of influence which forces the attacker to retreat.

It is frivolous, however to always maximize the attacker's cost because it may cause the attacker to engage the defender doing the exact opposite of what is intended. Minimizing the attacker's cost seems the only way to proceed, but that too can lead the attacker into a region where engagement of the defender is attractive. It is of critical importance that the defender not only minimizes the attacker's cost but also ensures that it follows a retreat trajectory carefully avoiding regions where engagement is posed as the most attractive option.

The conflict of interest in both of the agents can be analyzed in the domain of Game Theory. Differential Game theory was first introduced by Rufus Issacs [5] and has since been applied to scenarios such as the one described above. A two team differential game was analysed in [4]. In this paper, the attacker gets close to the high-value target before the mobile defender could intercept it.

The idea for this thesis was obtained from [3]. The paper presents a solution for an Engage or Retreat Game where the defender was immobile. The thesis proposes a two-player game -the attacker and the defender, both of which are mobile. The mobile attacker must choose between two possibilities-to attack the mobile defender or to retreat to a predefined boundary. The mobile defender on the other hand is to protect itself or any targets by persuading the attacker to retreat. To achieve this, the defender must either maximize the attacker's functional or minimize the attacker's utility.

The general technique solution used to solve the game has been taken from [2]. The equilibrium strategies for each agent is expressed as a solution of one of two related optimization problems. The initial position of the mobile agent acts determines which optimal solution should be implemented. The first optimization problem known as *GoE* in which the attacker captures the defender while the defender minimizes the utility function of the Attacker. The second Optimization problem is known as the *OCR*. In this, the attacker retreats to a pre-defined boundary called the retreat surface with the cooperation of the defender. The defender does this by maximizing the utility function of the attacker. To ensure that the equilibrium trajectories do not enter the region of engagement, an inequality constraint is imposed on the retreat trajectory with the help of the value function from the Game of Engagement. Regions of constrained retreat called *escort regions* are produced where the attacker is escorted away by the defender from engagement regions. It is crucial to note that the GoE and OCR are related problems whose solutions are proven to be equilibrium solutions to the ERG as a function of the initial state of the mobile agents.

The thesis begins with explaining the Engage or Retreat Game in Section 2. The solutions to the game are developed in Section 3. The Game of Engagement and Optimal Constrained Retreat are solved in Section 3.1 and 3.2 respectively. The overall solution is presented in Section 3.3. Numerical examples that support the solutions developed in Section 3 are presented in section 4. Section 5 provides a brief conclusion of the thesis.

Game Description

The Engage or Retreat game consists of two mobile agents, the attacker and the defender, both of whom by manipulating the state of the system strive to maximize their respective utility functionals. The choice to engage the defender or retreat lies with the attacker. The defender in order to escape capture, tries to maximize or minimize the attacker's utility.

2.1 System Description

For ease of solving the game, two different but interchangeable coordinate systems are used to describe the state of the system. The first coordinate system,which we will refer to as the *global* coordinate system, is utilized to plot the agent trajectories and in certain instances to help justify the obtained results. In this system, the position of the attacker and the defender is defined by a pair of cartesian coordinates. The position of the attacker A is described by the Cartesian Coordinates $\mathbf{x}_A = (x_A, y_A)$. Similarly, the position of the defender D is described by the Cartesian Coordinates $\mathbf{x}_D = (x_D, y_D)$. The state of the system can be completely represented by the vector $\mathbf{x}_G = (x_A, y_A, x_D, y_D)$.

Both the attacker and defender move with constant speed which are defined as v_A for the attacker and v_D for the defender. The attacker controls its heading through its control variable ψ, and the defender controls its heading through its control variable θ. In the global coordinate system, both headings are measured counterclockwise from the x-

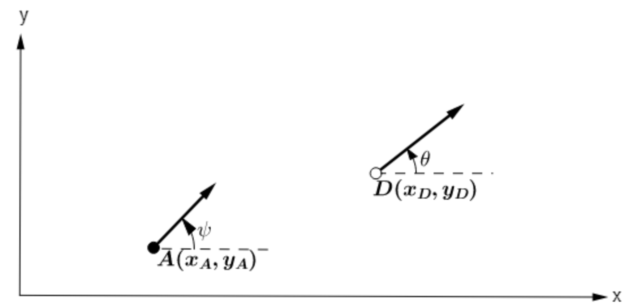

Figure 2.1: Global Co-ordinate system

axis. The defender possesses an additional control variable, ϕ, which appears within the attacker's utility function. The attacker's control, u_A, and the defender's control, $\mathbf{u}_D$, are defined as

$$u_A := \psi \tag{2.1}$$

$$u_D := (\theta, \phi) \tag{2.2}$$

The global coordinate system is visually depicted in Figure 2.1.

Using the global coordinate system, the behavior of the system is completely described by a system of four ordinary differential equations:

$$\begin{pmatrix} \dot{x}_A \\ \dot{y}_A \\ \dot{x}_D \\ \dot{y}_D \end{pmatrix} = \begin{pmatrix} v_A cos\psi \\ v_A sin\psi \\ v_D cos\theta \\ v_D sin\theta \end{pmatrix} = \mathbf{f}(\mathbf{x}_G, u_A, \mathbf{u}_D) \tag{2.3}$$

The second coordinate system, called as the *relative* coordinate system, is utilized in order to simplify the development of optimality conditions. In this, the location of the defender is represented with respect to the attacker. This system will reduce the number of dimensions and thereby make it easier to calculate the equilibrium solutions. Once computed, the results shall be converted to the global coordinate system and plotted.

The state of the system is completely represented by $\mathbf{x}_R = (d, \beta, x_A, y_A)$. The first state component d is the distance between the attacker and defender. The second state component β is the angle made by the defender with respect to the attacker measured in the counter-clockwise direction from the positive direction of the x axis. The third and fourth state components, x_A and y_A, represent the position of the attacker. Figure 2.2 shows a graphical representation of the relative system dynamics. The global and relative coordinates are related using the following equations.

$$d = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2} \tag{2.4}$$

$$\beta = tan\left(\frac{y_D - y_A}{x_D - x_A}\right) \tag{2.5}$$

The control variables of the relative and global coordinates are related accordingly:

$$\psi = \hat{\psi} + \beta \tag{2.6}$$

$$\theta = \hat{\theta} + \beta \tag{2.7}$$

From the Figure 2.2 define $\left(\frac{x_D - x_A}{d}\right) = cos(\beta)$ and $\left(\frac{y_D - y_A}{d}\right) = sin(\beta)$. When equation

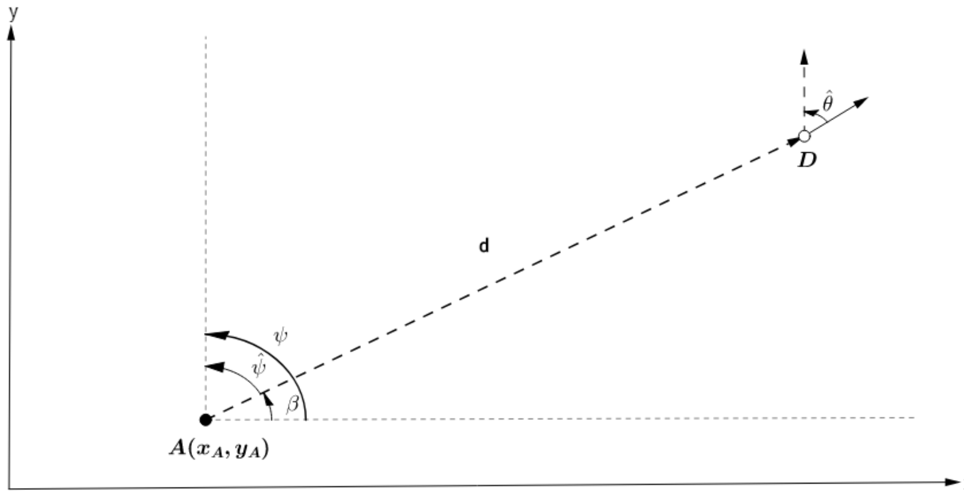

Figure 2.2: Relative Co-ordinate System

(2.4) is differentiated with respect to time we get,

$$\begin{aligned}
\frac{d(d)}{dt} &= \frac{d}{dt}\sqrt{(x_D - x_A)^2 + (y_D - y_A)^2} \\
&= \frac{2(x_D - x_A)(\dot{x}_D - \dot{x}_A) + 2(y_D - y_A)(\dot{y}_D - \dot{y}_A)}{2\sqrt{(x_D - x_A)^2 + (y_D - y_A)^2}} \\
&= \frac{2(x_D - x_A)(\dot{x}_D - \dot{x}_A) + 2(y_D - y_A)(\dot{y}_D - \dot{y}_A)}{d} \\
&= (-v_A cos\psi + v_D cos\theta)cos\beta + (-v_A sin\psi + v_D sin\theta)sin\beta \\
&= -v_A cos\hat{\psi} + v_D cos\hat{\theta} \qquad (2.8)
\end{aligned}$$

Similarly, when equation(2.5) is differentiated with respect to time we get,

$$
\begin{aligned}
\frac{d(\beta)}{dt} =& \frac{d}{dt}tan\left(\frac{y_D - y_A}{x_D - x_A}\right) \\
=& \frac{1}{1 + (\frac{y_D - y_A}{x_D - x_A})^2}\frac{d}{dt}\frac{(y_D - y_A)}{(x_D - x_A)} \\
=& \frac{(x_D - x_A)^2}{(x_D - x_A)^2 + (y_D - y_A)^2}\left(\frac{(x_D - x_A)(\dot{y}_D - \dot{y}_A)}{(x_D - x_A)^2} - \frac{(y_D - y_A)(\dot{x}_D - \dot{x}_A)}{(x_D - x_A)^2}\right) \\
=& \frac{cos(\beta)(v_D sin\theta - v_A sin\psi)}{d} - \frac{sin\beta(v_D cos\theta - v_A cos\psi)}{d} \\
=& \frac{v_D sin\hat{\theta} - v_A sin\hat{\psi}}{d} \qquad (2.9)
\end{aligned}
$$

Therefore, the reduced space kinematic equations are:

$$\dot{x}_A = v_A cos(\hat{\psi} + \beta) \qquad (2.10)$$

$$\dot{y}_A = v_A sin(\hat{\psi} + \beta) \qquad (2.11)$$

$$\dot{d} = -v_A cos\hat{\psi} + v_D cos\hat{\theta} \qquad (2.12)$$

$$\dot{\beta} = \frac{v_D sin\hat{\theta} - v_A sin\hat{\psi}}{d} \qquad (2.13)$$

To achieve the defender's coordinates from the relative coordinate solution the following conversion is used. From Figure(2.2) it can be seen that

$$x_A = x_A \qquad (2.14)$$

$$y_A = y_A \qquad (2.15)$$

$$x_D = dcos\beta + x_A \qquad (2.16)$$

$$y_D = dsin\beta + y_A \qquad (2.17)$$

.

The termination conditions play a vital role in determining when the Engage or Retreat game ends or terminates. The vector $\mathbf{x}_f$ denotes the state at the terminal time t_f. The terminal time is defined as the point at which the state of the system satisfies either the engagement condition $\Gamma_E(\mathbf{x}) = 0$ or the retreat condition $\Gamma_R(\mathbf{x}) = 0$. The capture condition is satisfied when the attacker comes within the capture distance, $d_c > 0$, and the defender:

$$\Gamma_E(\mathbf{x}_f) = d - d_c = 0. \tag{2.18}$$

The retreat condition is met when the attacker retreats a defined retreat boundary:

$$\Gamma_R(\mathbf{x}_f) = y_A - y_R = 0. \tag{2.19}$$

The sets of state values that satisfy the terminal conditions engagement surface, $\mathbf{X}_E$, and retreat surface, $\mathbf{X}_R$, can now be defined with the help of their respective terminal conditions:

$$\begin{aligned}\mathbf{X}_E &: = \{\mathbf{x} \in \mathbb{R}^4 | \Gamma_E(\mathbf{x}) = 0\} \\ \mathbf{X}_R &: = \{\mathbf{x} \in \mathbb{R}^4 | \Gamma_R(\mathbf{x}) = 0\}.\end{aligned}$$

2.2 Agent Utilities

Both agents strive to maximize their utility functions by manipulating the state of the system. Although the utility functions depend on the initial state $\mathbf{x}_0$, it is not an independent variable and cannot be controlled by either agent. Hence, it will be included as a parameter and not as an independent variable in the utility functions. The attacker's utility function consists of a terminal value function and an integral cost function. The defender's utility function consists of the terminal value function alone. The utility function of the attacker and the utility function of the defender are defined as:

$$U_A(u_A(t), \mathbf{u}_D(t); \mathbf{x}_0) := \phi_A(\mathbf{x}_f) - \int_{t_0}^{t_f} C_A(u_A(\tau), \mathbf{u}_D(\tau), \mathbf{x}(\tau)) d\tau \tag{2.20}$$

$$U_D(u_A(t), \mathbf{u}_D(t); \mathbf{x}_0) := \phi_D(\mathbf{x}_f) \tag{2.21}$$

In equation (2.20), the function $C_A(u_A(\tau), \mathbf{u}_D(\tau), \mathbf{x}(\tau))$ is the instantaneous cost that shall be integrated over the duration of the game and $\phi_A(\mathbf{x}_f)$ is the terminal value function for the attacker. The instantaneous cost function is defined as

$$C_A(u_A(t), \mathbf{u}_D(t), \mathbf{x}(t)) = -(\phi + c_2) \quad \text{where} \quad 0 \leq \phi \leq c_1 \tag{2.22}$$

The constant c_2 represents a time or energy penalty for the mobile attacker. The function $\phi_D(\mathbf{x}_f)$ is the terminal value function for the defender.

Each agent has a preference as to whether the game must terminate in engagement or retreat. Suppose, the attacker prefers that the game terminates in engagement over retreat if the integral cost is ignored.

$$\phi_A(\mathbf{x}_E) > \phi_A(\mathbf{x}_R) \quad \forall \quad \mathbf{x}_E \in \mathbf{X}_E, \mathbf{x}_R \in \mathbf{X}_R \tag{2.23}$$

If the choice were to be made by the defender, preference would be given to terminating the game in retreat over engagement.

$$\phi_D(\mathbf{x}_E) = -b_1 < \phi_D(\mathbf{x}_R) = b_2 \quad \forall \quad \mathbf{x}_E \in \mathbf{X}_E, \mathbf{x}_R \in \mathbf{X}_R \tag{2.24}$$

where b_1 and b_2 are constants. The reason they are constant for all $\mathbf{x} \in \mathbf{X}_E$ and $\mathbf{x} \in \mathbf{X}_R$ is because the sole concern of the defender is whether to terminate the game in engagement or retreat. The terminal value functions are:

$$\phi_A(\mathbf{x}_f) = \begin{cases} a_1, & \mathbf{x}_f \in \mathbf{X}_E \\ 0, & \mathbf{x}_f \in \mathbf{X}_R \end{cases} \tag{2.25}$$

$$\phi_D(\mathbf{x}_f) = \begin{cases} -b_1, & \mathbf{x}_f \in \mathbf{X}_E \\ 0, & \mathbf{x}_f \in \mathbf{X}_R \end{cases} \tag{2.26}$$

It is assumed that $a_1 > 0$, $b_1 > 0$. An assumption is made that each agent has complete knowledge of the state of the system for the entire duration of the game. Another assumption made is that the attacker exercises superior control and dynamic characteristics in such a manner that it can force the game to terminate in either from the engagement region $\mathbf{x}_E \in \mathbf{X}_E$ or in the retreat region $\mathbf{x}_R \in \mathbf{X}_R$ from any initial state $\mathbf{x}_0 \in \mathbf{R}_A$. Therefore, the attacker has the freedom to choose whether to end the game in engagement or retreat and the defender cannot deter the attacker or prevent the attacker directly from engaging it.

The defender indirectly deters the attacker from engaging it, by manipulating the instantaneous cost of the game present in the utility function of the attacker as seen in equation(2.20). The manipulation is done in such a manner that a trajectory that leads from the initial state $\mathbf{x}_0$ to the retreat state $\mathbf{x}_R$ yields a better utility value than all the trajectories that would travel from the initial state $\mathbf{x}_0$ to any state in the engagement region $\mathbf{x}_E \in \mathbf{X}_E$. If the defender can successfully create the above stated scenario, the attacker will choose to retreat instead of engaging the defender, thereby maximizing the defender's utility function because $\phi_D(\mathbf{x}_E) < \phi_D(\mathbf{x}_R)$.

2.3 Game Definition

A differential game is defined with the agent utility functions (2.20) and (2.21) as defined above:

$$U_A^*(\mathbf{u}_D(t);\mathbf{x}_0) = \max_{\mathbf{u}_A(t)} U_A(u_A(t),\mathbf{u}_D(t);\mathbf{x}_0) \tag{2.27}$$

$$U_D^*(u_A(t);\mathbf{x}_0) = \max_{\mathbf{u}_D(t)} U_D(u_A(t),\mathbf{u}_D(t);\mathbf{x}_0) \tag{2.28}$$

For a given initial state, each agent attempts to maximize their respective utility functions. The Nash Equilibrium solution to the game defined above is the pair of equilibrium open-loop strategies $u_A^*(t;\mathbf{x}_0)$ and $\mathbf{u}_D^*(t;\mathbf{x}_0)$, and the resulting equilibrium utility values $U_A^*(\mathbf{x}_0)$ and $U_D^*(\mathbf{x}_0)$ that satisfy the below listed Nash Equilibrium conditions:

$$U_A(u_A(t),\mathbf{u}_D^*(t);\mathbf{x}_0) = U_A^*(\mathbf{x}_0) \geq U_A(u_A(t),\mathbf{u}_D^*(t);\mathbf{x}_0) \tag{2.29}$$

$$U_D(u_A^*(t),\mathbf{u}_D^*(t);\mathbf{x}_0) = U_D^*(\mathbf{x}_0) \geq U_D(u_A^*(t),\mathbf{u}_D(t);\mathbf{x}_0) \tag{2.30}$$

Solution Technique

The solution to the *Engage or Retreat Game* is obtained by addressing two related optimization problems referred to as the Game of Engagement(GoE) and Optimal Constrained Retreat(OCR). The solution to either the GoE or OCR for any given initial state $\mathbf{x}_0$ will be the equilibrium solution to the general ERG. The strategies for each agent are expressed in terms of solutions to one of the two optimization problems.

3.1 Game of Engagement

The first optimization problem called the *Game of Engagement or Game of Attack* represents the situation when the attacker engages/attacks the defender. It has already been mentioned that the attacker possesses the choice to force the state to the termination surface. The game terminates when the attacker captures the defender *i.e;*, when the attacker reaches within a pre-defined distance radius of the defender known as capture distance.

In the game of engagement or game of attack, the attacker attempts to terminate the game in engagement *i.e;* maximize the utility function of the attacker while the defender minimizes the utility function of the attacker U_A. The attacker gets a bonus or reward a_1 whereas the defender pays a penalty.

Using the system dynamics and the objective of the agents, the differential game is defined as

$$V_E(\mathbf{x}_0) := \min_{u_D(t)} \max_{u_A(t)} U_A(u_A(t), \mathbf{u}_D(t), \mathbf{x}_0) \tag{3.1}$$

with the constraint that $\Gamma_E(\mathbf{x}_f) = 0$.

The equilibrium value of the game starting at the initial conditions $\mathbf{x}_0$ is represented by the function $V_E(\mathbf{x}_0)$ when the attacker and the defender implement their respective equilibrium open-loop strategies $u_A^{E*}(t;\mathbf{x}_0)$ and $\mathbf{u}_D^{E*}(t;\mathbf{x}_0)$ where

$$u_A^{E*}(t;\mathbf{x}_0), \mathbf{u}_D^{E*}(t;\mathbf{x}_0) = \arg\max_{u_A(t)} \min_{\mathbf{u}_D(t)} U_A(u_A(t), \mathbf{u}_D(t), \mathbf{x}_0) \tag{3.2}$$

This type of formulation represents what is called a standard pursuit evasion game of degree.

The solution to the game is obtained using standard differential techniques. Firstly, the Hamiltonian is constructed using the system dynamics. Secondly, the adjoint variables are defined and the optimal control laws are derived. Thirdly, a lagrangian multiplier is found by solving the Hamiltonian at terminal time. Lastly, the terminal time and value function along with the state trajectories are derived.

3.1.1 Solution in Global Coordinate System

The Hamiltonian is constructed using the system dynamics (2.3) and the cost function (2.22) to determine the solution to the game of engagement. The terminal reward or bonus obtained at the end of the Game of Engagement is not dependent on time and thereby renders the Hamiltonian to be equal to zero along the optimal trajectories.

$$
\begin{aligned}
\mathbf{H}_E :&= \boldsymbol{\lambda}_E^T \mathbf{f}(\mathbf{x}, u_A, \mathbf{u}_D) + C_T(\mathbf{x}, u_A, \mathbf{u}_D) \\
&= \lambda_{xAE}\dot{x}_A + \lambda_{YAE}\dot{y}_A + \lambda_{xDE}\dot{x}_D + \lambda_{yDE}\dot{y}_D + C_T \\
&= \lambda_{xAE}(v_A cos\psi) + \lambda_{yAE}(v_A sin\psi) + \\
&\lambda_{xDE}(v_D cos\theta) + \lambda_{yDE}(v_D sin\theta) + C_T \\
&= \lambda_{xAE}(v_A cos\psi) + \lambda_{yAE}(v_A sin\psi) + \\
&\lambda_{xDE}(v_D cos\theta) + \lambda_{yDE}(v_D sin\theta) - (\phi + c_2) = 0
\end{aligned}
\tag{3.3}
$$

The vector $\boldsymbol{\lambda}_E$ contains the adjoint variables conjugate to the kinematics of the attacker and defender and is also equal to the gradient of the value function.

$$
\boldsymbol{\lambda}_E := (\lambda_{xAE}, \lambda_{yAE}, \lambda_{xDE}, \lambda_{yDE})^T = \left(\frac{\partial V_E}{\partial x_{AE}}, \frac{\partial V_E}{\partial y_{AE}}, \frac{\partial V_E}{\partial x_{DE}}, \frac{\partial V_E}{\partial y_{DE}}\right)^T
$$

The adjoint equations are found by taking the partial derivative of the Hamiltonian with respect to each of the state components:

$$
\dot{\lambda}_E := -\frac{\partial H_E}{\partial \mathbf{x}}^T = \left(-\frac{\partial H_E}{\partial x_{AE}}, -\frac{\partial H_E}{\partial y_{AE}}, -\frac{\partial H_E}{\partial x_{DE}}, -\frac{\partial H_E}{\partial y_{DE}}\right)^T
$$

$$
\dot{\lambda}_{xAE} = -\frac{\partial H}{\partial x_{AE}} = 0 \tag{3.4}
$$

$$
\dot{\lambda}_{yAE} = -\frac{\partial H}{\partial y_{AE}} = 0 \tag{3.5}
$$

$$
\dot{\lambda}_{xDE} = -\frac{\partial H}{\partial x_{DE}} = 0 \tag{3.6}
$$

$$
\dot{\lambda}_{yDE} = -\frac{\partial H}{\partial y_{DE}} = 0 \tag{3.7}
$$

An adjoined terminal value function is constructed using the capture condition (2.18):

$$\begin{aligned}\Phi_E(\mathbf{x}_f) &= a_1 + \vartheta_E \Gamma_E(x) \\ \Phi_E(\mathbf{x}_f) &= a_1 + \vartheta_E \left(\sqrt{(x_{Af}^2 - x_{Df}^2) + (y_{Af}^2 - y_{Df}^2)} - d_c \right)\end{aligned} \tag{3.8}$$

where ϑ_E is a Lagrangian multiplier and (x_{Af}, y_{Af}) represents the position of the attacker at terminal time and (x_{Df}, y_{Df}) represents the position of the defender at terminal time. The adjoint variables at terminal time are obtained by taking the partial derivative of the terminal value function ϕ_E with respect to the appropriate state variables:

$$\boldsymbol{\lambda}_E(t_f) := \frac{\partial \phi_E}{\partial \mathbf{x}_f} = \left(\frac{\partial \phi_E}{\partial x_{AE}}, \frac{\partial \phi_E}{\partial y_{AE}}, \frac{\partial \phi_E}{\partial x_{DE}}, \frac{\partial \phi_E}{\partial y_{DE}} \right)^T \tag{3.9}$$

$$\lambda_{xAE}(t_f) = \frac{\partial \phi_E(x)}{\partial x_{AE}} = \frac{\partial}{\partial x_{AE}} [\vartheta_E \Gamma_E(x)] \tag{3.10}$$

$$\begin{aligned}\lambda_{xAE}(t_f) &= \vartheta_E \frac{\partial}{\partial x_{AE}} \left(\sqrt{(x_{Af}^2 - x_{Df}^2) + (y_{Af}^2 - y_{Df}^2)} \right) \\ &= \vartheta_E \frac{2(x_{Af} - x_{Df})}{2\sqrt{(x_{Af}^2 - x_{Df}^2) + (y_{Af}^2 - y_{Df}^2)}} \\ &= \frac{\vartheta_E (x_{Af} - x_{Df})}{d_c}\end{aligned} \tag{3.11}$$

Similarly,

$$\lambda_{yAE}(t_f) = \frac{\vartheta_E (y_{Af} - y_{Df})}{d_c} \tag{3.12}$$

Similarly for the defender,

$$\lambda_{xDE}(t_f) = \frac{-\vartheta_E(x_{Af} - x_{Df})}{d_c} \tag{3.13}$$

$$\lambda_{yDE}(t_f) = \frac{-\vartheta_E(y_{Af} - y_{Df})}{d_c} \tag{3.14}$$

The equilibrium control for the attacker and defender is obtained by minimizing and maximizing the Hamiltonian respectively. The Hamiltonian H (3.3) is partially differentiated with respect to the attacker and defender control(heading angle).

$$\frac{\partial H_E}{\partial \psi^{E*}} = 0$$

$$\lambda_{xAE}(v_A sin\psi^{E*}) + \lambda_{yAE}(v_A cos\psi^{E*}) = 0$$

$$tan\psi^{E*} = \frac{\lambda_{yAE}}{\lambda_{xAE}}$$

$$cos\psi^{E*} = \frac{-\lambda_{xAE}}{\sqrt{\lambda_{xAE}^2 + \lambda_{yA}^2}} \tag{3.15}$$

$$sin\psi^{E*} = \frac{-\lambda_{yAE}}{\sqrt{\lambda_{xAE}^2 + \lambda_{yAE}^2}} \tag{3.16}$$

Similarly,

$$\frac{\partial H_E}{\partial \theta^{E*}} = 0$$

$$\lambda_{XDE}(v_D sin\theta^{E*}) + \lambda_{yD}(v_D cos\theta^{E*}) = 0$$

$$tan\theta^{E*} = \frac{\lambda_{yDE}}{\lambda_{xDE}}$$

$$cos\theta^{E*} = \frac{\lambda_{xDE}}{\sqrt{\lambda_{xDE}^2 + \lambda_{yDE}^2}} \tag{3.17}$$

$$sin\theta^{E*} = \frac{\lambda_{yDE}}{\sqrt{\lambda_{xDE}^2 + \lambda_{yDE}^2}} \tag{3.18}$$

Therefore, the equilibrium control for the attacker and defender are given by equations (3.15) to and (3.18) respectively. We solve for ϑ_E by substituting the optimal control equations(3.15) to (3.18) in the Hamiltonian equation (3.3) and evaluating at terminal time:

$$
\begin{aligned}
\mathbf{H}_E := \lambda_{xAE}(t_f)&\frac{\lambda_{xAE}(t_f)}{\sqrt{\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f)}}v_A \\
+\lambda_{yAE}(t_f)&\frac{\lambda_{yAE}(t_f)}{\sqrt{\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f)}}v_A \\
+\lambda_{xDE}(t_f)&\frac{-\lambda_{xDE}(t_f)}{\sqrt{\lambda^2_{xDE}(t_f)+\lambda^2_{yDE}(t_f)}}v_D \\
+\lambda_{yDE}(t_f)&\frac{-\lambda_{yDE}(t_f)}{\sqrt{\lambda^2_{xDE}(t_f)+\lambda^2_{yDE}(t_f)}}v_D \\
&-(\phi+c_2)=0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H}_E :&= \frac{\left(\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f)\right)}{\sqrt{\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f)}}v_A - \frac{\left(\lambda^2_{xDE}(t_f)+\lambda^2_{yAE}(t_f)\right)}{\sqrt{\lambda^2_{xDE}(t_f)+\lambda_{yDE}(t_f)^2(t_f)}}v_D - (\phi+c_2)=0 \\
\mathbf{H}_E :&= \left(\sqrt{\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f)}\right)v_A - \left(\sqrt{\lambda^2_{xDE}(t_f)+\lambda^2_{yDE}(t_f)}\right)v_D - (\phi+c_2)=0
\end{aligned}
\tag{3.19}
$$

From (3.11) and (3.12) the following relation is obtained:

$$
\begin{aligned}
\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f) &= \frac{\vartheta^2_E(x_{AE}-x_{DE})^2}{d_c^2}+\frac{\vartheta^2_E(y_{AE}-y_{DE})^2}{d_c^2} \\
\lambda^2_{xAE}(t_f)+\lambda^2_{yAE}(t_f) &= \frac{\vartheta^2_E}{d_c^2}\left((x_{AE}-x_{DE})^2+(y_{AE}-y_{DE})^2\right)
\end{aligned}
\tag{3.20}
$$

Using equation (2.4) and (2.18) in equation (3.20),

$$
\begin{aligned}
\lambda_{xAE}^2(t_f) + \lambda_{yAE}^2(t_f) &= \frac{\vartheta_E^2(d_c^2)}{d_c^2} \\
\lambda_{xAE}^2(t_f) + \lambda_{yAE}^2(t_f) &= \vartheta_E^2
\end{aligned}
\tag{3.21}
$$

Similarly using equations (3.13) and (3.14):

$$
\lambda_{xDE}^2 + \lambda_{yDE}^2 = \vartheta_E^2 \tag{3.22}
$$

Using equations (3.20) and (3.22) in equation (3.19):

$$
v_A(\vartheta_E^2) - v_D(\vartheta_E^2) - (c_1 + c_2) = 0
$$

Rearranging to obtain ϑ_E:

$$
\begin{aligned}
\vartheta_E^2 &= \frac{c_1 + c_2}{v_A - v_D} \\
|\vartheta_E| &= \frac{c_1 + c_2}{v_A - v_D}
\end{aligned}
\tag{3.23}
$$

This implies that ϑ_E could be positive or negative. We choose the negative value because at terminal time, so that the attacker moves towards the defender.

$$
\vartheta_E = -\frac{c_1 + c_2}{v_A - v_D} \tag{3.24}
$$

The adjoint values at the terminal time as a function of the terminal state are obtained by substituting equation (3.24) in the terminal adjoint equations (3.11) to (3.14):

$$\lambda_{xAE}(t_f) = -\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{x_{AE} - x_{DE}}{d_c}\right) \tag{3.25}$$

$$\lambda_{yAE}(t_f) = -\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{y_{AE} - y_{DE}}{d_c}\right) \tag{3.26}$$

$$\lambda_{xDE}(t_f) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{x_{AE} - x_{DE}}{d_c}\right) \tag{3.27}$$

$$\lambda_{yDE}(t_f) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{y_{AE} - y_{DE}}{d_c}\right) \tag{3.28}$$

A boundary value problem is created by combining the optimal control equations(3.15) to (3.18), system dynamics(3.33) to(3.36), adjoint equations (3.4) to (3.7) and terminal adjoint equations (3.25) to (3.28). The complete solution to the game can be calculated analytically for each set of initial state $\boldsymbol{x_0}$. The solution to the boundary value problem is the solution to the differential game of engagement.

Theorem 1: Suppose the differential game is initiated at $\mathbf{x_0} = (x_{A0}, y_{A0}, x_{D0}, y_{D0})$*.The equilibrium control strategies are,*

$$u_A^{E*}(t; \mathbf{x_0}) = (\psi^{E*}(t; \mathbf{x_0})) \tag{3.29}$$

$$\mathbf{u}_D^{E*}(t; \mathbf{x_0}) = (\theta^{E*}(t; \mathbf{x_0}), \phi^{E*}(t; \mathbf{x_0})) \tag{3.30}$$

The resulting state trajectories for $\mathbf{x_0}$ are as follows:

$$\phi^{E*}(t;\mathbf{x_0}) = c_1, \tag{3.31}$$

$$\psi^{E*}(t;\mathbf{x_0}) = \gamma_E, \theta^{E*}(t;\mathbf{x_0}) = \delta_E \tag{3.32}$$

$$x_A^{E*}(t;\mathbf{x_0}) = v_A cos\gamma_E t + x_{A0} \tag{3.33}$$

$$y_A^{E*}(t;\mathbf{x_0}) = v_A sin\gamma_E t + y_{A0} \tag{3.34}$$

$$x_D^{E*}(t;\mathbf{x_0}) = v_D cos\delta_E t + x_{D0} \tag{3.35}$$

$$y_D^{E*}(t;\mathbf{x_0}) = v_D sin\delta_E t + y_{D0} \tag{3.36}$$

$$t_f^*(\mathbf{x_0}) = \frac{\sqrt{(x_{A0}-x_{D0})^2 + (y_{A0}-y_{D0})^2} - d_c}{v_A - v_D} \tag{3.37}$$

$$V_E(\mathbf{x_0}) = a_1 - \left(\frac{c_1+c_2}{v_A-v_D}\right)\left(\sqrt{(x_{A0}-x_{D0})^2 + (y_{A0}-y_{D0})^2} - d_c\right) \tag{3.38}$$

where

$$cos\gamma_E = \frac{(x_{AE}-x_{DE})}{\sqrt{(x_{AE}-x_{DE})^2 + (y_{AE}-y_{DE})^2}} \tag{3.39}$$

$$sin\gamma_E = \frac{(y_{AE}-y_{DE})}{\sqrt{(x_{AE}-x_{DE})^2 + (y_{AE}-y_{DE})^2}} \tag{3.40}$$

$$cos\delta_E = -\frac{(x_{AE}-x_{DE})}{\sqrt{(x_{AE}-x_{DE})^2 + (y_{AE}-y_{DE})^2}} \tag{3.41}$$

$$sin\delta_E = -\frac{(y_{AE}-y_{DE})}{\sqrt{(x_{AE}-x_{DE})^2 + (y_{AE}-y_{DE})^2}} \tag{3.42}$$

Proof: Since the adjoint derivatives (3.4) to (3.7) do not change over the entire time(they are zero), the adjoint variables λ_{xA}^{E*} ,λ_{yA}^{E*},λ_{xD}^{E*} and λ_{yD}^{E*} are constant as shown below.

$$\lambda_{xA}^{E*}(t) = -\left(\frac{c_1+c_2}{v_A-v_D}\right)\left(\frac{(x_{AE}-x_{DE})}{\sqrt{(x_{AE}-x_{DE})^2 + (y_{AE}-y_{DE})^2}}\right) \tag{3.43}$$

Similarly,

$$\lambda_{yA}^{E*}(t) = -\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{y_{AE} - y_{DE}}{\sqrt{(x_{AE} - x_{DE})^2 + (y_{AE} - y_{DE})^2}}\right) \tag{3.44}$$

$$\lambda_{xD}^{E*}(t) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{x_{AE} - x_{DE}}{\sqrt{(x_{AE} - x_{DE})^2 + (y_{AE} - y_{DE})^2}}\right) \tag{3.45}$$

$$\lambda_{yD}^{E*}(t) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\frac{y_{AE} - y_{DE}}{\sqrt{(x_{AE} - x_{DE})^2 + (y_{AE} - y_{DE})^2}}\right) \tag{3.46}$$

Combining equations (3.43) to (3.46) with the equilibrium control found for the attacker and the defender (3.15) to (3.18), it can be determined that $\psi^{E*}(t, \mathbf{x_0})$ and $\theta^{E*}(t, \mathbf{x_0})$ are constant throughout the entire game of engagement.The optimal control strategies can be written as,

$$\begin{aligned} cos(\psi^{E*}) &= -\frac{\vartheta_E\left(\frac{x_{Af} - x_{Df}}{d_c}\right)}{\sqrt{\vartheta_E^2\frac{(x_{Af}-x_{Df})^2}{d_c^2} + \vartheta_E^2\frac{(y_{Af}-y_{Df})^2}{d_c^2}}} \\ &= -\frac{(x_{Af} - x_{Df})}{\sqrt{(x_{Af} - x_{Df})^2 + (y_{Af} - y_{Df})^2}} \\ &= -\frac{(x_{Af} - x_{Df})}{d_c} \end{aligned} \tag{3.47}$$

Similarly,

$$sin(\psi^{E*}) = -\frac{(y_{Af} - y_{Df})}{d_c} \tag{3.48}$$

$$cos(\theta^{E*}) = \frac{(x_{Af} - x_{Df})}{d_c} \tag{3.49}$$

$$sin(\theta^{E*}) = \frac{(y_{Af} - y_{Df})}{d_c} \tag{3.50}$$

We then integrate the equations (3.47) to (3.50) and (2.3) backwards in time to generate the optimal trajectories and calculate the terminal time and the Value function.

$$\int_{t_0}^{t_f} \dot{x_A} = \int_{t_0}^{t_f} (v_A cos\psi) dt$$
$$x_A^{E*}(t;\mathbf{x}_0) = v_A cos\psi \int_{t_0}^{t_f} 1dt$$
$$x_A^{E*}(t;\mathbf{x}_0) = v_A cos\gamma_E t + x_{A0} \tag{3.51}$$

Similarly,

$$y_A^{E*}(t;\mathbf{x_0}) = v_A sin\gamma_E t + y_{A0} \tag{3.52}$$
$$x_D^{E*}(t;\mathbf{x_0}) = v_D cos\delta_E t + x_{D0} \tag{3.53}$$
$$y_D^{E*}(t;\mathbf{x_0}) = v_D sin\delta_E t + y_{D0} \tag{3.54}$$

The optimal strategies (3.47) to (3.50) at terminal time are substituted in the system dynamics (3.33) to (3.36) which give:

$$x_{AE}(t_f) = \frac{v_A(x_{Af} - x_{Df})t_f}{d_c} + x_{A0} \tag{3.55}$$
$$y_{AE}(t_f) = \frac{v_A(y_{Af} - y_{Df})t_f}{d_c} + y_{A0} \tag{3.56}$$
$$x_{DE}(t_f) = \frac{v_D(x_{Af} - x_{Df})t_f}{d_c} + x_{D0} \tag{3.57}$$
$$y_{DE}(t_f) = \frac{v_D(y_{Af} - y_{Df})t_f}{d_c} + y_{D0} \tag{3.58}$$

Rearranging equations (3.55) to (3.58):

$$x_{A0} = x_{AE}(t_f) - \frac{v_A(x_{Af} - x_{Df})t_f}{d_c} \tag{3.59}$$

$$y_{A0} = y_{AE}(t_f) - \frac{v_A(y_{Af} - y_{Df})t_f}{d_c} \tag{3.60}$$

$$x_{D0} = x_{DE}(t_f) - \frac{v_D(x_{Af} - x_{Df})t_f}{d_c} \tag{3.61}$$

$$y_{D0} = y_{DE}(t_f) - \frac{v_D(y_{Af} - y_{Df})t_f}{d_c} \tag{3.62}$$

Substituting equations (3.59) to (3.62) in the capture condition (2.18) to solve for terminal time:

$$(x_{A0} - x_{D0})^2 \left(1 - \frac{t_f(v_A - v_D)}{d_c}\right)^2 + (y_{A0} - y_{D0})^2 \left(1 - \frac{t_f(v_A - v_D)}{d_c}\right)^2 - d_c = 0 \tag{3.63}$$

Rearranging for terminal time,

$$t_f^*(\mathbf{x_0}) = \frac{\sqrt{(x_{A0} - x_{D0})^2 + (y_{A0} - y_{D0})^2} - d_c}{v_A - v_D}$$

The Value function for the game of engagement is obtained from the terminal time.

$$V_E(\mathbf{x_0}) = a_1 - \int_{t_0}^{t_f} (c_1 + c_2)$$

$$V_E(\mathbf{x_0}) = a_1 - (c_1 + c_2)(t_f - t_0)$$

$$V_E(\mathbf{x_0}) = a_1 - \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(\sqrt{(x_{A0} - x_{D0})^2 + (y_{A0} - y_{D0})^2} - d_c\right)$$

When these equilibrium strategies are implemented, the resulting utility functionals for the agents are:

$$U_A(\mathbf{x}_0)^{E*} := U_A(u_A^*(t;\mathbf{x}_0), \mathbf{u}_D^*(t;\mathbf{x}_0);\mathbf{x}_0) = V_E(\mathbf{x}_0) \tag{3.64}$$

$$U_D(\mathbf{x}_0)^{E*} := U_D(u_A^*(t;\mathbf{x}_0), \mathbf{u}_D^*(t;\mathbf{x}_0);\mathbf{x}_0) = \phi_D(\mathbf{x}_f \in \mathbf{X}_E) = b_1 \tag{3.65}$$

3.1.2 Solution in Relative Coordinate System

The *Game of Engagement* is solved in the Relative Coordinate system in the same manner as was followed for the Global Coordinate system. The optimal control strategies and the resulting trajectories have to be calculated in order to solve the game. Firstly, the optimality conditions of differential games are calculated as set forth first in Rufus Isaacs. The Hamiltonian for the equations is constructed as follows by utilising the system dynamics (2.10)-(2.13) and the cost function (2.22) as follows:

$$\begin{aligned}
H_E =& \boldsymbol{\lambda}_E^T \mathbf{f}(\mathbf{x}_R, u_A, \mathbf{u}_D) + C_T(\mathbf{x}_R, u_A, \mathbf{u}_D) \\
H_E =& \lambda_{xAE}\dot{x_A} + \lambda_{yAE}\dot{y_A} + \lambda_{\beta E}\dot{\beta} + \lambda_{dE}\dot{d} - (\phi + c_2) \\
=& \lambda_{xAE}(v_A cos(\hat{\psi} + \beta)) + \lambda_{yAE}(v_A sin(\hat{\psi} + \beta)) \\
&+ \lambda_{\beta E}\left(\frac{-v_A sin\hat{\psi} + v_D sin\hat{\theta}}{d}\right) \\
&+ \lambda_{dE}(-v_A cos\hat{\psi} + v_D cos\hat{\theta}) - (\phi + c_2) \\
&= 0
\end{aligned} \tag{3.66}$$

The game of engagement ends in capture and the reward for ending the game is not dependent on time, the Hamiltonian will be zero along the optimal trajectories. The vector λ_E contains the adjoint variable conjugate to the kinematics, the adjoint equations are found by taking the partial derivative of the hamiltonian(3.66) with respect to the corresponding state components.

$$\boldsymbol{\lambda}_E := (\lambda_{xAE}, \lambda_{yAE}, \lambda_{\beta E}, \lambda_{dE})^T = \left(\frac{\partial V_E}{\partial x_{AE}}, \frac{\partial V_E}{\partial y_{AE}}, \frac{\partial V_E}{\partial \beta_E}, \frac{\partial V_E}{\partial d_E}\right)^T \tag{3.67}$$

The adjoint equations are found by taking the partial derivative of the Hamiltonian with respect to each of the state components:

$$\dot{\lambda}_E := -\frac{\partial H_E}{\partial \mathbf{x}}^T = \left(-\frac{\partial H_E}{\partial x_{AE}}, -\frac{\partial H_E}{\partial y_{AE}}, -\frac{\partial H_E}{\partial \beta_E}, -\frac{\partial H_E}{\partial d_E}\right)^T$$

$$\dot{\lambda}_{xAE} = -\frac{\partial H}{\partial x_{AE}} = 0 \tag{3.68}$$

$$\dot{\lambda}_{yAE} = -\frac{\partial H}{\partial y_{AE}} = 0 \tag{3.69}$$

$$\dot{\lambda}_{\beta E} = -\frac{\partial H}{\partial \beta_E} = \lambda_{xAE} v_A sin(\hat{\psi} + \beta) - \lambda_{yAE} v_A cos(\hat{\psi} + \beta) \tag{3.70}$$

$$\dot{\lambda}_{dE} = -\frac{\partial H}{\partial d_E} = \frac{\lambda_{\beta E}}{d^2}\left(v_D sin\hat{\theta} - v_A sin\hat{\psi}\right) \tag{3.71}$$

A terminal value function is constructed from the capture condition (2.18) with the terminal reward.

$$\Phi_E(\mathbf{x}_f) = a_1 + \vartheta_E \Gamma_E(x)$$

$$\Phi_E(\mathbf{x}_f) = a_1 + \vartheta_E(d - d_c) \tag{3.72}$$

where a_1 is the terminal reward.

Taking partial derivatives of the terminal condition (3.72) with respect to the appropriate state components, the terminal adjoint variables are obtained.

$$\begin{aligned}\lambda_{xAE}(t_f) &= \frac{\partial \phi_E}{\partial x_A} \\ &= \frac{\partial \vartheta_E(d-d_c)}{\partial x_A} \\ &= 0 \end{aligned} \tag{3.73}$$

Similarly, the remaining terminal adjoint equations are derived as:

$$\lambda_{yAE}(t_f) = \frac{\partial \phi_E}{\partial y_A} = 0 \tag{3.74}$$

$$\lambda_{\beta E}(t_f) = \frac{\partial \phi_E}{\partial \beta} = 0 \tag{3.75}$$

$$\lambda_{dE}(t_f) = \frac{\partial \phi_E}{\partial d} = \vartheta_E \tag{3.76}$$

The Hamiltonian is always zero along the optimal trajectories and at terminal time $t = t_f$ it can be solved to obtain a lagrangian multiplier ν_E with the help of the optimal control which will be obtained as follows. When the adjoint variables at terminal time (3.73) to (3.71) are substituted in the Hamiltonian (3.66) the following equation is obtained:

$$H_E(t_f) = \lambda_{dE}(t_f)[v_D cos(\hat{\theta}^{E*})(t_f) - v_A cos(\hat{\psi}^{E*})(t_f)] - (c_1 + c_2) = 0 \tag{3.77}$$

Using lemma on circular vectograms from Issacs[5], the attacker is maximized and the defender is minimized and thereby, the following equations are obtained.

$$\begin{aligned} cos\hat{\psi}^{E*}(t_f) &= \frac{-|\lambda_{dE}(t_f)|v_A}{\sqrt{(-\lambda_{dE}(t_f)v_A)^2}} \\ &= \frac{-|\lambda_{dE}(t_f)|}{\lambda_{dE}(t_f)} \\ &= -sgn(\lambda_{dE}(t_f)) \end{aligned} \tag{3.78}$$

A property of the signum function $\frac{x}{|x|} = sgn(x)$ is used. Similarly,

$$\begin{aligned} cos\hat{\theta}^{E*}(t_f) =& \frac{-|\lambda_{dE}(t_f)|v_D}{\sqrt{(-\lambda_{dE}(t_f)v_D)^2}} \\ =& \frac{-|\lambda_{dE}(t_f)|}{\lambda_{dE}(t_f)} \\ =& - sgn(\lambda_{dE}(t_f)) \end{aligned} \tag{3.79}$$

Substituting equations (3.76), (3.78) and (3.79) in (3.66):

$$\begin{aligned} \vartheta_E v_D(-sgn(\vartheta_E)) - \vartheta_E v_A(-sgn(\vartheta_E)) - (c_1 + c_2) = 0 \\ (v_A - v_D)\vartheta_E sgn(\vartheta_E) = (c_1 + c_2) \\ \vartheta_E sgn(\vartheta_E) = \left(\frac{c_1 + c_2}{v_A - v_D}\right) \end{aligned} \tag{3.80}$$

From the property of the signum function $x.sgn(x) = |x|$.

$$|\vartheta_E| = \left(\frac{c_1 + c_2}{v_A - v_D}\right)$$

For the game of engagement to terminate the distance between the attacker and defender should reduce with an increase in time. This is possible only when λ_{dE} is positive since $\dot{d}$ has to be negative. Therefore,

$$\vartheta_E = -\left(\frac{c_1 + c_2}{v_A - v_D}\right) \tag{3.81}$$

The lagrangian multiplier ϑ_E is taken as $-\left(\frac{c_1+c_2}{v_A-v_D}\right)$ because the attacker should capture the defender. So equation (3.81) is placed in: (3.78) and (3.79):

$$cos(\hat{\psi})^{E*}(t_f) = -sgn(\lambda_{dE}(t_f)) = -\frac{|\vartheta_E|}{\vartheta_E} = -\frac{\left|-\left(\frac{c_1+c_2}{v_A-v_D}\right)\right|}{-\left(\frac{c_1+c_2}{v_A-v_D}\right)} = 1 \tag{3.82}$$

$$cos(\hat{\theta})^{E*}(t_f) = -sgn(\lambda_{dE}(t_f)) = -\frac{|\vartheta_E|}{\vartheta_E} = -\frac{\left|-\left(\frac{c_1+c_2}{v_A-v_D}\right)\right|}{-\left(\frac{c_1+c_2}{v_A-v_D}\right)} = 1 \tag{3.83}$$

This leads to,

$$\hat{\psi}^{E*}(t_f) = arccos(1) = 0 \tag{3.84}$$

$$\hat{\theta}^{E*}(t_f) = arccos(1) = 0 \tag{3.85}$$

A boundary value problem is created by combining the optimal control strategies (3.84) and (3.85), adjoint equations (3.68) to (3.71), adjoint variables at terminal time (3.73) to (3.76) and system dynamics (2.10) to (2.13).

Theorem 2:Suppose the differential game of engagement begins at $\mathbf{x_0} = (x_{A0}, y_{A0}, d_0, \beta_0)$. *The equilibrium strategies are,*

$$u_A^{E*}(t; \mathbf{x}_0) = (\hat{\psi}^{E*}(t, \mathbf{x}_0)) \tag{3.86}$$

$$\mathbf{u}_D^{E*}(t; \mathbf{x}_0) = (\hat{\theta}^{E*}(t, \mathbf{x}_0), \phi^{E*}(t, \mathbf{x}_0)) \tag{3.87}$$

and the state trajectories are as follows:

$$\phi^{E*}(t;\mathbf{x}_0) = c_1, \tag{3.88}$$

$$\hat{\psi}^{E*}(t;\mathbf{x}_0) = 0, \hat{\theta}^{E*}(t;\mathbf{x}_0) = 0 \tag{3.89}$$

$$x_A^{E*}(t;\mathbf{x}_0) = v_A cos(\hat{\psi} + \beta)t + x_{A0} \tag{3.90}$$

$$y_A^{E*}(t;\mathbf{x}_0) = v_A sin(\hat{\psi} + \beta)t + y_{A0} \tag{3.91}$$

$$d^{E*}(t;\mathbf{x}_0) = (v_D cos\hat{\theta} - v_A cos\hat{\psi})t + d_0 \tag{3.92}$$

$$\beta^{E*}(t;\mathbf{x}_0) = \frac{(v_D sin\hat{\theta} - v_A sin\hat{\psi})t}{d} + \beta_0 \tag{3.93}$$

$$t_F^{E*}(\mathbf{x}_0) = \frac{d_0 - d_c}{v_A - v_D} \tag{3.94}$$

$$V_E(\mathbf{x}_0) = a_1 - \frac{(c_1 + c_2)(d_0 - d_c)}{(v_A - v_D)} \tag{3.95}$$

Proof: Since the adjoint equations (adjoint derivatives) (3.68) and (3.69) are zero and the adjoint variables at terminal time (3.73) and (3.74) are zero we know that:

$$\lambda_{xA}^{E*}(t) = 0 \tag{3.96}$$

$$\lambda_{yA}^{E*}(t) = 0 \tag{3.97}$$

Using equations (3.96) and (3.97) in (3.70) we know that:

$$\lambda_{\beta}^{E*}(t) = 0 \tag{3.98}$$

Using equation (3.98) in equation (3.71) we know that:

$$\lambda_d^{E*}(t) = \nu_E$$
$$\lambda_d^{E*}(t) = -\left(\frac{c_1 + c_2}{v_A - v_D}\right) \tag{3.99}$$

Combining equations (3.96) to (3.99) and the equilibrium control of the attacker (3.84) and the defender (3.85) it can be deduced that $\hat{\psi}^{E*}(t, \mathbf{x}_0) = 0$ and $\hat{\theta}^{E*}(t, \mathbf{x}_0) = 0$. We integrate backwards in time to get the optimal trajectories, the terminal time and the Value Function.

The distance d can be calculated from equation (2.12):

$$\int_{t_0}^{t_f} \dot{d}dt = \int_{t_0}^{t_f} (v_D cos\hat{\theta} - v_A cos\hat{\psi})dt \tag{3.100}$$

After integrating we get,

$$\begin{aligned} d^{E*}(t; \mathbf{x}_0) &= (v_D cos\hat{\theta} - v_A cos\hat{\psi})t + d(t_0) \\ d^{E*}(t; \mathbf{x}_0) &= (v_D cos\hat{\theta} - v_A cos\hat{\psi})t + d_0 \end{aligned} \tag{3.101}$$

where $d(t_0) = d_0$ is the distance at time t_0.

β can be derived from (2.13).

$$\int_{t_0}^{t_f} \dot{\beta} = \int_{t_0}^{t_f} \frac{(v_D sin(\hat{\theta}) - v_A sin(\hat{\psi})}{d} + ct \tag{3.102}$$

After integrating we get,

$$\beta^{E*}(t; \mathbf{x}_0) = \frac{(v_D sin\hat{\theta} - v_A sin\hat{\psi})t}{d} + ct \tag{3.103}$$

where $\beta(t_0) = \beta_0$ at time t_0.

$$\beta^{E*}(t;\mathbf{x}_0) = \frac{(v_D sin\hat{\theta} - v_A sin\hat{\psi})t}{d} + \beta(t_0)$$
$$\beta^{E*}(t;\mathbf{x}_0) = \frac{(v_D sin\hat{\theta} - v_A sin\hat{\psi})t}{d} + \beta_0 \tag{3.104}$$

Similarly,

$$\begin{aligned}\int_{t_0}^{t_f} \dot{x}_A dt = \int_{t_0}^{t_f} v_A cos(\hat{\psi} + \beta)dt \\ x_A(t_f) =& v_A cos(\hat{\psi} + \beta)\int_{t_0}^{t_f} 1dt \\ =& v_A cos(\hat{\psi} + \beta)t + ct\end{aligned}$$

where ct is a constant of integration.

At $t = t_0$ we have $x_A = x_A(t_0) = x_{A0}$. Therefore, we obtain

$$x_A^{E*}(t;\mathbf{x}_0) = v_A cos(\hat{\psi} + \beta)t + x_{A0} \tag{3.105}$$

Similarly,

$$y_A^{E*}(t;\mathbf{x}_0) = v_A sin(\hat{\psi} + \beta)t + y_{A0} \tag{3.106}$$

When the optimal control laws (3.84) and (3.85) are substituted in equation(3.92), the terminal time equation is obtained.

$$\begin{aligned} d(t_f) &= (v_D - v_A)t_F + d(t_0) \\ t_f(\mathbf{x}_0) &= \frac{-(d(t_f) - d(t_0))}{v_A - v_D} \\ t_f(\mathbf{x}_0) &= \frac{d_0 - d_c}{v_A - v_D} \end{aligned} \tag{3.107}$$

where $d(t_F) = d_c$ because the distance at terminal time is equal to the capture distance. The Value function for the Game of Engagement is obtained by integrating the cost over the period of time. The terminal time equation is obtained from (3.94).

$$\begin{aligned} V_E(\mathbf{x}_0) &= a_1 - \int_{t_0}^{t_f} (c_1 + c_2) \\ V_E(\mathbf{x}_0) &= a_1 - (c_1 + c_2)(t_f - t_0) \\ V_E(\mathbf{x}_0) &= a_1 - \frac{(c_1 + c_2)(d_0 - d_c)}{(v_a - v_d)} \end{aligned} \tag{3.108}$$

When these equilibrium strategies are implemented, the resulting utility functionals for the agents are:

$$U_A^{E*}(\mathbf{x}_0) := U_A(u_A^*(t;\mathbf{x}_0), \mathbf{u}_D^*(t;\mathbf{x}_0); \mathbf{x}_0) = V_E(\mathbf{x}_0) \tag{3.109}$$

$$U_D^{E*}(\mathbf{x}_0) := U_D(u_A^*(t;\mathbf{x}_0), \mathbf{u}_D^*(t;\mathbf{x}_0); \mathbf{x}_0) = \phi_D(\mathbf{x}_f \in \mathbf{X}_E) = b_1 \tag{3.110}$$

3.2 Optimal Constrained Retreat

The second optimization problem referred to as the *Optimal Constrained Retreat* arises when the attacker along with the co-operation of the defender, retreats to a pre-defined boundary. The attacker ends the game in retreat while attempting to maximizing its utility function. The defender also attempts to maximize the attacker's utility function. Additionally, the defender encourages the attacker to avoid moving into regions where engagement would be optimal.

Since both agents have the same objective of maximizing the attacker's utility function, they can be combined to form a single player and the problem reduces to a traditional continuous optimal control problem defined as:

$$\begin{aligned} V_R(\mathbf{x_0}) :&= \max_{u_A(t)} \max_{\mathbf{u}_D(t)} U_A(u_A(t), \mathbf{u}_D(t); \mathbf{x}_0) \\ &= \max_{u_A(t), u_D(t)} U_A(u_A(t), \mathbf{u}_D(t); \mathbf{x}_0) \end{aligned} \tag{3.111}$$

with the terminal constraint (2.19). During the retreat, it is plausible that the state may move into a region where engagement becomes an attractive option. To prevent the occurrence of such an incident and to ensure that the attacker goes into the retreat region, a constraint is imposed on the value function of the retreat.

$$V_R(\mathbf{x}(t)) - V_E(\mathbf{x}(t)) \geq 0 \quad \forall [t_0, t_f] \tag{3.112}$$

where $V_E(\mathbf{x}(t))$ is the value function for the Game Of engagement. To include the value function constraint (3.112) in the optimal control problem, we will create the additional state component $c(t)$ by converting it into a state inequality constraint. The time derivative of the additional state component $\dot{c}(t)$ is defined as:

$$\begin{aligned} \dot{c} &= -C_A(u_A(t), \mathbf{u}_D(t), \mathbf{x}(t)) \\ &= -(u_D c_1 + c_2) \end{aligned} \tag{3.113}$$

The state component signifies the remaining integral cost for the rest of the game and it has a terminal value of:

$$c(t_f) = 0 \tag{3.114}$$

The value function constraint is restated as a state variable constraint using the state component(3.114). The value function for the optimal constrained retreat $V_R(\mathbf{x})$ can now be rewritten with $c(t)$ as:

$$\begin{aligned} V_R(\mathbf{x}) &= \phi_A(\mathbf{x}(t_f)) + \int_t^{t_f} \dot{c}(\tau)d\tau \\ &= \phi_A(\mathbf{x}(t_f)) + \int_t^{t_f} C_A(u_A(\tau), \mathbf{u}_D(\tau), \mathbf{x}(\tau))d\tau \\ &= \phi_A(\mathbf{x}(t_f)) - c(t_f) + c(t) \\ &= \phi_A(\mathbf{x}(t_f)) + c(t) \end{aligned} \tag{3.115}$$

Using equation (3.115), the state inequality constraint (3.112) can be written as:

$$g(\mathbf{x}) = V_R(\mathbf{x}(t)) - V_E(\mathbf{x}(t)) \geq 0 \tag{3.116}$$

$$g(\mathbf{x}) = \phi_A(\mathbf{x}(t_f)) + c(t) - V_E(\mathbf{x}(t)) \geq 0 \tag{3.117}$$

The presence of a control variable is necessary to determine the effects of the constraint on the optimal control strategies. The state inequality constraint $g(\mathbf{x})$ is not an explicit function of control. It is differentiated with respect to time, till an expression that is solely dependent on the control of the attacker and defender.

$$h(\mathbf{x}) := \frac{d}{dt}g(\mathbf{x})$$

The control constraint is obtained by taking the derivative of the state inequality constraint

(3.116) which can be written as:

$$h(\mathbf{x}) := g'(\mathbf{x}) = V_R'(\mathbf{x}(t)) - V_E'(\mathbf{x}(t)) \tag{3.118}$$

The derivatives $V_R'(\mathbf{x}(t))$:

$$\begin{aligned} V_R'(\mathbf{x}(t)) &= \frac{d}{dt}\left[\int_0^{t_f} -(u_D c_1 + c_2)dt\right] \\ &= \frac{d}{dt}(-(c_2) * (t_f - t_o)) \\ &= \frac{d}{dt}(-(c_2) * (t_f) + (c_2) * (t_o)) \\ &= c_2 \end{aligned} \tag{3.119}$$

and $V_E'(\mathbf{x}(t))$:

$$\begin{aligned} V_E'(\mathbf{x}(t)) &= \frac{d}{dt}\left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right] \\ &= 0 - \left(\frac{c_1 + c_2}{v_A - v_D}\right)\dot{d} \end{aligned}$$

The dynamics of the state $\dot{d}$ is substituted in the above equation:

$$V_E'(\mathbf{x}(t)) = -\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(-v_A cos\hat{\psi} + v_D cos\hat{\theta}\right) \tag{3.120}$$

Substituting (3.119) and (3.120) into (3.119) the following is the control constraint $h(\mathbf{x})$:

$$\begin{aligned} h(\mathbf{x}) &= g'(\mathbf{x}) \\ &= c_2 + \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(-v_A cos\hat{\psi} + v_D cos\hat{\theta}\right) \end{aligned} \tag{3.121}$$

The Hamiltonian for the Optimal Constrained Retreat is constructed using the system dynamics (2.10) to (2.13), utility function of the attacker (2.20) and the control equality constraint (3.121) now becomes:

$$H_R = \boldsymbol{\lambda}_R(\dot{\mathbf{x}}) + \mu h(\mathbf{x}) + C_A(u_A(t), \mathbf{u}_D(t), \mathbf{x}(t))$$

Expanding the Hamiltonian:

$$\begin{aligned}
H_R = \lambda_{xAR}\dot{x}_A + \lambda_{yAR}\dot{y}_A + \lambda_{\beta R}\dot{\beta} + \lambda_{dR}\dot{d} + \lambda_{cR}\dot{c} + \mu h(\mathbf{x}) + [-(u_D c_1 + c_2)] \quad &= 0 \\
= \lambda_{xAR} v_A cos(\hat{\psi} + \beta) & \\
+ \lambda_{yAR} v_A sin(\hat{\psi} + \beta) & \\
+ \frac{\lambda_{\beta R}}{d}\left(v_D sin\hat{\theta} - v_A sin\hat{\psi}\right) & \\
+ \lambda_{dR}\left(v_D cos\hat{\theta} - v_A cos\hat{\psi}\right) & \\
+ \lambda_{cR}[-(u_D c_1 + c_2)] & \\
+ \mu\left[-(u_D c_1 + c_2) + \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(-v_A cos\hat{\psi} + v_D cos\hat{\theta}\right)\right] & \\
+ [-(u_D c_1 + c_2)] & \\
= 0 &
\end{aligned} \tag{3.122}$$

where λ_R is the adjoint variable that contains the gradient of the value function of the OCR:

$$\boldsymbol{\lambda}_R := (\lambda_{xAR}, \lambda_{yAR}, \lambda_{\beta R}, \lambda_{dR}, \lambda_{cR})^T = \left(\frac{\partial V_R}{\partial x_{AR}}, \frac{\partial V_R}{\partial y_{AR}}, \frac{\partial V_R}{\partial \beta_R}, \frac{\partial V_R}{\partial d_R}, \frac{\partial V_R}{\partial c_R}\right)^T \tag{3.123}$$

The additional adjoint variable μ is a scalar and seeks to impose the control constraint when the state inequality constraint becomes active displaying the behaviour listed below.

$$\mu(t) = 0 \quad g(\mathbf{x}(t)) > 0$$
$$\mu(t) > 0 \quad g(\mathbf{x}(t)) = 0$$

The adjoint equations are found by taking the partial derivative of the Hamiltonian (3.122) with respect to the appropriate state components. The General form of the adjoint equations that effectively, represents the piece-wise behaviour for the constrained and unconstrained arcs.

$$\dot{\lambda}_R = \frac{\partial H_R}{\partial \mathbf{x}} \begin{cases} \lambda_R \dfrac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} + \mu \dfrac{\partial h}{\partial \mathbf{x}} + \dfrac{\partial C}{\partial \mathbf{x}} & g(\mathbf{x}) = 0 \\ \lambda_R \dfrac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} + \dfrac{\partial C}{\partial \mathbf{x}} & g(\mathbf{x}) \geq 0 \end{cases}$$

However, in this Optimal Constrained Retreat problem,the control constraint (3.121), does not have any state components so the additional adjoint variable does not appear directly in the adjoint equations.

$$\dot{\lambda}_{xAR} = -\frac{\partial H_R}{\partial x_{AR}} = 0 \tag{3.124}$$
$$\dot{\lambda}_{yAR} = -\frac{\partial H_R}{\partial y_{AR}} = 0 \tag{3.125}$$
$$\dot{\lambda}_{\beta R} = -\frac{\partial H_R}{\partial \beta_R} = \lambda_{xAR} v_A sin(\hat{\psi} + \beta) - \lambda_{yAR} v_A cos(\hat{\psi} + \beta) \tag{3.126}$$
$$\dot{\lambda}_{dR} = -\frac{\partial H_R}{\partial d_R} = \frac{\lambda_{\beta R}}{d^2} \left(v_D sin\hat{\theta} - v_A sin\hat{\psi} \right) \tag{3.127}$$
$$\dot{\lambda}_{cR} = 0 \tag{3.128}$$

A terminal value function is constructed with the terminating condition (2.19) where ν_R is a lagrangian multiplier.

$$\phi_R(\mathbf{x}_f) = 0 + \nu_R \Gamma_R(\mathbf{x})$$
$$\phi_R(\mathbf{x}_f) = 0 + \nu_R(y_A - y_R) \tag{3.129}$$

The values of the adjoint variables at terminal time are calculated by taking the partial derivative of (3.129) with respect to the appropriate state component:

$$\boldsymbol{\lambda}_R(t_f) := \frac{\partial \Phi_R}{\partial \mathbf{x}_f}^T = \left(\frac{\partial \Phi_R}{\partial x_{AR}}, \frac{\partial \Phi_R}{\partial y_{AR}}, \frac{\partial \Phi_R}{\partial \beta_R}, \frac{\partial \Phi_R}{\partial d_R}, \frac{\partial \Phi_R}{\partial c_R} \right)^T \tag{3.130}$$

$$\lambda_{xAR}(t_f) = \frac{\partial \phi_R}{\partial x_{AR}} = 0 \tag{3.131}$$
$$\lambda_{yAR}(t_f) = \frac{\partial \phi_R}{\partial y_{AR}} = \nu_R(1 - 0) = \nu_R \tag{3.132}$$
$$\lambda_{\beta R}(t_f) = \frac{\partial \phi_R}{\partial \beta R} = 0 \tag{3.133}$$
$$\lambda_{dR}(t_f) = \frac{\partial \phi_R}{\partial d_R} = 0 \tag{3.134}$$
$$\lambda_{cR}(t_f) = \frac{\partial \phi_R}{\partial c_R} = 0 \tag{3.135}$$

The Hamiltonian is rearranged so that the co-efficients of the attacker and defender optimal control can be easily taken.

$$
\begin{aligned}
cos\hat{\psi}^{R*}\left(\lambda_{xAR}v_A cos\beta + \lambda_{yAR}v_A sin\beta - \lambda_{dR}v_A - \mu\left(\frac{c_1+c_2}{v_A-v_D}\right)v_A\right) \\
+cos\hat{\theta}^{R*}\left(\lambda_{dR}v_D + \mu\left(\frac{c_1+c_2}{v_A-v_D}\right)v_D\right) \\
+sin\hat{\psi}^{R*}\left(-\lambda_{xAR}v_A sin\beta + \lambda_{yAR}v_A cos\beta - \frac{\lambda_{\beta R}}{d}v_A\right) \\
+sin\hat{\theta}^{R*}\left(\frac{\lambda_{\beta R}d}{v_D}\right) \\
+\left(\lambda_{cR}+\mu+1\right)\left[-(u_D c_1 + c_2)\right] = 0
\end{aligned}
\tag{3.136}
$$

The optimal control for each agent is calculated by maximizing the Hamiltonian (3.122),

$$cos\hat{\psi}^{R*} = \frac{k_1}{\rho_A} \tag{3.137}$$

$$cos\hat{\theta}^{R*} = \frac{k_3}{\rho_D} \tag{3.138}$$

$$sin\hat{\psi}^{R*} = \frac{k_2}{\rho_A} \tag{3.139}$$

$$sin\hat{\theta}^{R*} = \frac{k_4}{\rho_D} \tag{3.140}$$

where

$$
\begin{aligned}
k_1 &= \lambda_{xAR} v_A cos\beta + \lambda_{yAR} v_A sin\beta \\
&\quad - \lambda_{dR} v_A - \mu \left(\frac{c_1 + c_2}{v_A - v_D} \right) v_A && (3.141) \\
k_2 &= -\lambda_{xAR} v_A sin\beta + \lambda_{yAR} v_A cos\beta - \frac{\lambda_{\beta R}}{d} v_A && (3.142) \\
k_3 &= \lambda_{dR} v_D + \mu \left(\frac{c_1 + c_2}{v_A - v_D} \right) v_D && (3.143) \\
k_4 &= \frac{\lambda_{\beta R} d}{v_D} && (3.144) \\
\rho_A &= \sqrt{k_1^2 + k_2^2} && (3.145) \\
\rho_D &= \sqrt{k_3^2 + k_4^2} && (3.146)
\end{aligned}
$$

k_1, k_2, k_3 and k_4 are the co-efficients of $cos\hat{\psi}^{R*}, sin\hat{\psi}^{R*}, cos\hat{\theta}^{R*}$ and $sin\hat{\theta}^{R*}$ respectively. The resulting Defender cost control:

$$
u_D = \begin{cases} 0, & \text{for } -\lambda_{cR} - \mu - 1 \leq 0 \\ 1, & \text{for } -\lambda_{cR} - \mu - 1 > 0 \end{cases}
$$

The game is assumed to end on an unconstrained segment thereby rendering the adjoint variable $\mu = 0$. The lagrangian multiplier ν_R is calculated by evaluating the Hamiltonian(3.122) at terminal time, the adjoint equations (3.131) to (3.135) at terminal time and equilibrium control strategies (3.137) to (3.140):

$$
\begin{aligned}
H^{R*}|_{t=t_f} &= cos\hat{\psi}^{R*}(\lambda_{yAR} v_A sin\beta) + sin\hat{\psi}^{R*}(\lambda_{yAR} v_A cos\beta) + (u_D c_1 + c_2)(-\lambda_{cR} - \mu - 1) = 0 \\
&= \frac{k_1}{\rho_A}(\lambda_{yAR} v_A sin\beta) + \frac{k_2}{\rho_A}(\lambda_{yAR} v_A cos\beta) - c_2 = 0 \\
&= \frac{\lambda_{yAR}^2 v_A^2}{|\lambda_{yAR}| v_A} - c_2 = 0
\end{aligned}
\tag{3.147}
$$

Solving for $\lambda_{yAR} = \nu_R$, gives:

$$|\nu_R| = \frac{c_2}{v_A} \tag{3.148}$$

To decide the sign of the parameter, we look at the attacker's equilibrium control which makes $\nu_R = -\frac{c_2}{v_A}$. Also, looking at the value function $V_R(\mathbf{x}_0)$, $\lambda_{yAR} = \frac{\partial V_R}{\partial y_{AR}} = -\frac{c_2}{v_A}$. This implies that the cost will be integrated over a longer period of time which results in a lower resulting value.

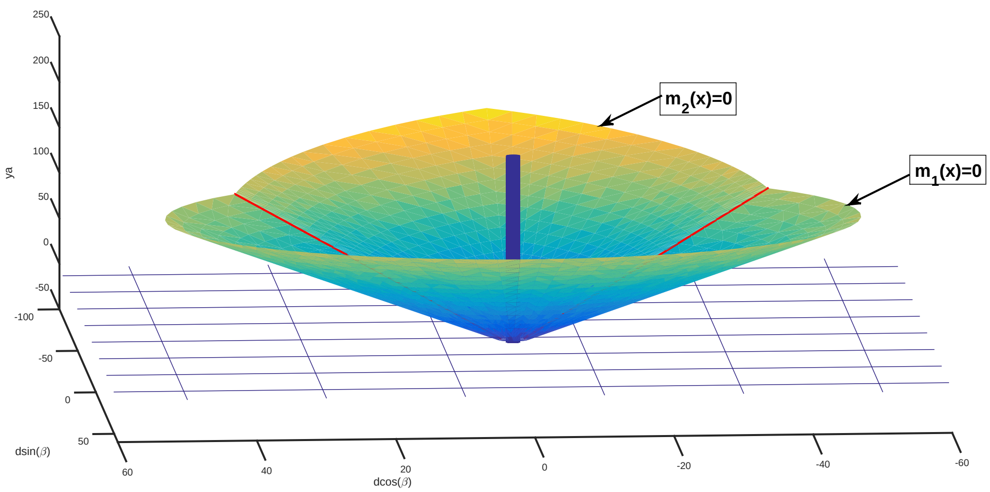

Figure 3.1: Solution Regions for OCR

The structure of the optimal trajectories for certain initial states is complicated by the state inequality constraint (3.116) imposed. Additionally, a solution does not exist that satisfies the value constraint imposed by (3.117) for certain other initial states $\mathbf{x}_0$. All possible initial states $\mathbf{x}_0 = (x_{A0}, y_{A0}, d_0, \beta_0)$ fall into any of the four regions defined below as shown in Figure 3.1:

$$\mathbf{R}_1 := \{\mathbf{x}_0 \in \mathbf{R}_A | \beta < \beta_{T1}, \beta > \beta_{T2}\} \tag{3.149}$$

$$\mathbf{R}_2 := \{\mathbf{x}_0 \in \mathbf{R}_A | \beta_{T1} < \beta < \beta_{T2}\} \tag{3.150}$$

$$\mathbf{R}_3 := \{\mathbf{x}_0 \in \mathbf{R}_A | \beta_{T1} < \beta < \beta_{T2}, m_1(\mathbf{x}_0) < 0, 0 < m_2(\mathbf{x}_0)\} \tag{3.151}$$

$$\mathbf{R}_4 := \{\mathbf{x}_0 \in \mathbf{R}_A | \beta_{T1} < \beta < \beta_{T2}, m_2(\mathbf{x}_0) \leq 0\} \tag{3.152}$$

where:

$$m_1(\mathbf{x}) = -\frac{c_2(y_{AT} - y_R)}{v_A} - a_1 + (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right) \tag{3.153}$$

$$y_{AT} = \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right]\frac{v_A}{c_2} + y_R \tag{3.154}$$

$$\beta_{T1} = asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \tag{3.155}$$

$$\beta_{T2} = \pi - asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \tag{3.156}$$

The proof for equations (3.154),(3.155) and (3.156) are shown in Theorem 4. An equation could not be obtained by hand for $m_2(\mathbf{x}) = 0$. It will be explained further on how the solution for that region was obtained.

Theorem 3:The system dynamics for the unconstrained segment or a given initial con-

dition $\mathbf{x}_0 = (x_{A0}, y_{A0}, d_0, \beta_0)$ *at time* t_0 *for region* $R_1 \cup R_2$

$$\hat{\psi}^{R*}(t;\mathbf{x}_0) = -\frac{\pi}{2} - \beta \tag{3.157}$$

$$\hat{\theta}^{R*}(t;\mathbf{x}_0) = triviallyoptimal \tag{3.158}$$

$$x_A^{R*}(t;\mathbf{x}_0) = x_A(t_f) \tag{3.159}$$

$$y_A^{R*}(t;\mathbf{x}_0) = y_R + (t - t_f)v_A \tag{3.160}$$

$$t_f^{R*}(\mathbf{x}_0) = t_0 + \frac{y_A - y_R}{v_A} \tag{3.161}$$

$$V_R^{R*}(\mathbf{x}_0) = -\frac{c_2}{v_A}(y_A - y_R) \tag{3.162}$$

Proof: Starting on the terminal surface,defined by equation (2.19) at terminal time based on the assumption that the game terminates on an unconstrained segment thereby, rendering $\mu = 0$. The attacker control would be best calculated as follows:

$$\begin{aligned} tan(\hat{\psi}_f^{R*}) &= \left.\frac{k_2}{k_1}\right|_{t=t_f} \\ &= \left.\frac{-\lambda_{xAR}v_A sin\beta + \lambda_{yAR}v_A cos\beta - \frac{\lambda_{\beta R}}{d}v_A}{\lambda_{xAR}v_A cos\beta + \lambda_{yAR}v_A sin\beta - \lambda_{dR}v_A - \mu\left(\frac{c_1+c_2}{v_A-v_D}\right)v_A}\right|_{t=t_f} \end{aligned} \tag{3.163}$$

When the terminal equations, (3.131) to (3.134) are substituted in (3.163), the optimal control for the attacker for the unconstrained segment at terminal time becomes:

$$
\begin{aligned}
tan(\hat{\psi}_f^{R*}) &= \left.\frac{\lambda_{yAR}v_A cos\beta}{\lambda_{yA}v_A sin\beta}\right|_{t=t_f} \\
&= \left.\frac{-cos\beta}{-sin\beta}\right|_{t=t_f} \\
&= \pi - \left(\frac{\pi}{2} + \beta\right) \\
&= -\frac{\pi}{2} - \beta
\end{aligned} \tag{3.164}
$$

The adjoint variables at terminal time (3.131) to (3.134) when substituted in equations (3.200) and (3.201) becomes as follows:

$$
\begin{aligned}
k_3 &= 0 \\
k_4 &= 0 \\
\rho_D &= 0
\end{aligned}
$$

$$
cos\hat{\theta}^{R*} = \frac{k_3}{\rho_D} = \frac{0}{0} \tag{3.165}
$$

$$
sin\hat{\theta}^{R*} = \frac{k_4}{\rho_D} = \frac{0}{0} \tag{3.166}
$$

From equations (3.165) and (3.166) the defender's control is trivially optimal as long as it remains in the above defined region.

By examining the adjoint variables at terminal time(3.131),(3.132) ,(3.135) and the adjoint equation derivatives (3.124),(3.125),(3.128), the adjoint equations for the duration of unconstrained segment are deduced.

$$
\lambda_{xAR}(t) = 0 \tag{3.167}
$$

$$
\lambda_{yAR}(t) = -\frac{c_2}{v_A} \tag{3.168}
$$

$$
\lambda_{cR}(t) = 0 \tag{3.169}
$$

Using the optimal control derived as per eqns(3.164) ,(3.182), the adjoint variables at terminal time(3.133),(3.134), the adjoint equations (3.126),(3.127) along with the adjoint equations for the duration of unconstrained segment (3.167),(3.168) and (3.169) the equations reduce as follows:

$$\begin{aligned}\dot{\lambda}_{\beta R} &= 0 + \frac{c_2}{v_A} cos(-\frac{\pi}{2} - \beta + \beta) \\ &= 0 \end{aligned} \tag{3.170}$$

This makes

$$\dot{\lambda}_{dR} = 0 \tag{3.171}$$

Thereby,

$$\lambda_{\beta R}(t) = 0 \tag{3.172}$$

$$\lambda_{dR}(t) = 0 \tag{3.173}$$

Substituting equations (3.167) to (3.173) the attacker optimal control for the unconstrained segment is obtained as,

$$sin\psi^{R*}(t) = -1 \tag{3.174}$$

$$cos\psi^{R*}(t) = 0 \tag{3.175}$$

Theorem 4: At the point of exit $\mathbf{x}_T = (d_T, \beta_T, x_{AT}, y_{AT})$*, the set of state values that satisfy* *the tangency constraints are given by:*

$$y_{AT} = \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right]\frac{v_A}{c_2} + y_R \tag{3.176}$$

$$\beta_{T1} = asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \tag{3.177}$$

$$\beta_{T2} = \pi - asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \tag{3.178}$$

Proof: The points where the optimal trajectory exits the constrained arc are called tangency points. Tangency conditions come into play when the state constraint becomes active. These must be imposed to compensate for the constant term lost in the differentiation. The tangency lines are calculated by starting at terminal time on the terminal surface (2.19) and integrating backwards in time so that a parametric solution can be obtained for the unconstrained segment. Define t_1 as the time where the state constraint becomes active, and when the optimal trajectory exits the unconstrained segment. At the point of exit, the equilibrium trajectories must satisfy both the constraints.

The equations (3.116) and (3.121) are the tangency conditions and defined as:

$$N(\mathbf{x}(t_1)) = \begin{bmatrix} g(\mathbf{x}(t_1)) \\ g'(\mathbf{x}(t_1)) \end{bmatrix} = 0$$

The state constraint and its derivative are:

$$\boxed{g(\mathbf{x}) = -\frac{c_2(y_A - y_R)}{v_A} - a_1 + \left(\frac{c_1 + c_2}{v_A - v_D}\right)(d - d_c)}$$

$$\boxed{h(\mathbf{x}) = c_2 + \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(v_D cos\hat{\theta} - v_A cos\hat{\psi}\right)}$$

Rearranging the equations to obtain the tangency conditions in terms of the state dynamics:

Taking $g(x)$ equation (3.116) and substituting (3.95) and (3.162) in it:

$$g(x) = -\frac{c_2(y_{AT} - y_R)}{v_A} - \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right] = 0$$

$$a_1 + \frac{c_2(y_{AT} - y_R)}{v_A} - \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right] = 0$$

$$\frac{c_2(y_{AT} - y_R)}{v_A} = -a_1 + \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right]$$

$$y_{AT} = \left[a_1 - (c_1 + c_2)\left(\frac{d - d_c}{v_A - v_D}\right)\right]\frac{v_A}{c_2} + y_R \tag{3.179}$$

It can be seen from equations (3.165) and (3.166) the optimal control of the defender is indeterminate. In order to solve the game, it is necessary to take the first step of integration. It is imperative to know the optimal control of the defender. The limit of evader's control is used as t approaches t_1 which is the time when the state constraint becomes active.

$$\begin{aligned}\lim_{t\to t_1}(tan\hat{\theta}^{R*}) &= \lim_{t\to t_1}\frac{k_4}{k_3} \\ &= \lim_{t\to t_1}\frac{\frac{\lambda_{\beta R} d}{v_D}}{\lambda_{dR} v_D + \mu\left(\frac{c_1+c_2}{v_A - v_D}\right) v_D} \\ &= \lim_{t\to t_1}\frac{0}{0}\end{aligned} \tag{3.180}$$

Since it still remains indeterminate, L'hospital's rule is applied to calculate the defender control.

$$
\begin{aligned}
\lim_{t \to t_1} (tan\hat{\theta}^{R*}) &= \lim_{t \to t_1} \frac{\dot{k}_4}{\dot{k}_3} \\
&= \frac{\frac{\dot{\lambda}_{\beta R} d - \lambda_{\beta R} \dot{d}}{v_D^2}}{\dot{\lambda}_{dR} v_D + \dot{\mu} \left(\frac{c_1 + c_2}{v_A - v_D} \right) v_D} \\
&= \frac{0}{\dot{\lambda}_{dR} v_D + \dot{\mu} \left(\frac{c_1 + c_2}{v_A - v_D} \right) v_D} \\
&= 0 \qquad (3.181)
\end{aligned}
$$

That means,

$$\lim_{t \to t_1} \theta^{\hat{R}*} = 0 \qquad (3.182)$$

This implies that the control of the defender is trivially optimal as long as it out of the retreat boundary.

Taking the second tangency condition $g'(\mathbf{x}) = h(\mathbf{x})$ and solving it:

$$(c_1 + c_2) \left(v_D cos\hat{\theta} - v_A cos\hat{\psi} \right) = (v_A - v_D) c_2 \qquad (3.183)$$

$$c_2 = - \left(\frac{c_1 + c_2}{v_A - v_D} \right) \left(v_D cos\hat{\theta} - v_A cos\hat{\psi} \right) \qquad (3.184)$$

Substituting the optimal control for the unconstrained segments from eqns (3.164) and

(3.182) the second tangency condition is rewritten as:

$$\begin{aligned}
c_2 &= -\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(v_D(1) - v_A cos\left(-\frac{\pi}{2}\right) - \beta_T\right) \\
c_2 &= -\left(\frac{c_1 + c_2}{v_A - v_D}\right)(v_D(1) + v_A sin\beta_T)
\end{aligned}$$

Rearranging the above equation in terms of state variable:

$$\begin{aligned}
sin\beta_T &= -\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A} \\
\beta_{T1} &= asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \qquad (3.185) \\
\beta_{T2} &= \pi - asin\left(\left[\frac{c_2}{v_A}\left(\frac{v_A - v_D}{c_1 + c_2}\right)\right] - \frac{v_D}{v_A}\right) \qquad (3.186)
\end{aligned}$$

The solution for $h(x)$ gives two solutions which leads to two tangency lines.

The tangency conditions are non-linear in nature. Moreover, the value function for the game of retreat when the constraint is active is not known due to the non-linear nature of y_A.

The value of state inequality constraint μ is to be obtained as follows. The value constraint eqn (3.183) is differentiated with respect to the appropriate state components and adjoint variables. Taking advantage of that, $\dot{\mu}$ is found as follows:

$$\dot{h}(x) = \frac{dh}{dX}\dot{X} + \frac{dh}{d\lambda}\dot{\lambda} + \frac{dh}{d\mu}\dot{\mu} \qquad (3.187)$$

$$\dot{\mu} = \frac{-\left(\frac{dh}{dX}\dot{X} + \frac{dh}{d\lambda}\dot{\lambda}\right)}{\frac{dh}{d\mu}} \qquad (3.188)$$

The eqn (3.187) when substituted with all the terms would be as follows:

$$\dot{h}(x) = \frac{\partial h}{\partial x_A}\dot{x}_A + \frac{\partial h}{\partial y_A}\dot{y}_A \tag{3.189}$$

$$+ \frac{\partial h}{\partial \beta}\dot{\beta} + \frac{\partial h}{\partial d}\dot{d} \tag{3.190}$$

$$+ \frac{\partial h}{\partial \lambda_{x_A}}\dot{\lambda}_{xA} + \frac{\partial h}{\partial \lambda_{y_A}}\dot{\lambda}_{yA} \tag{3.191}$$

$$+ \frac{\partial h}{\partial \lambda_\beta}\dot{\lambda}_\beta + \frac{\partial h}{\partial \lambda_d}\dot{\lambda}_d \tag{3.192}$$

$$+ \frac{\partial h}{\partial \lambda_c}\dot{\lambda}_c + \frac{\partial h}{\partial \mu}\dot{\mu} \tag{3.193}$$

When the optimal control (3.137) to (3.140) along with (3.198) to (3.146) for the constrained segment is substituted in $h(x)$. The control constraint $h(x)$ becomes:

$$h(x) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left[v_D cos\left(\arctan\left(\frac{k4}{k3}\right)\right) - v_A cos\left(\arctan\left(\frac{k2}{k1}\right)\right)\right] \tag{3.194}$$

The partial differentiation of $\dot{h}(x)$ with each state is as follows:

$$\begin{aligned}\frac{\partial h}{\partial z} = &\left(\frac{c_1 + c_2}{v_A - v_D}\right)\left[-v_D sin\left(\arctan\left(\frac{k_4}{k_3}\right)\right)\frac{k_3^2}{k_3^2 + k_4^2}\frac{k_3\frac{\partial k_4}{\partial z} - k_4\frac{\partial k_3}{\partial z}}{k_3^2}\right] \\ &+ \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left[v_A sin\left(\arctan\left(\frac{k_2}{k_1}\right)\right)\frac{k_1^2}{k_1^2 + k_2^2}\frac{k_2\frac{\partial k_1}{\partial z} - k_1\frac{\partial k_2}{\partial z}}{k_1^2}\right]\end{aligned} \tag{3.195}$$

where z stands for $d_R, \beta_R, x_{AR}, y_{AR}, c, \lambda_{xAR}, \lambda_{yAR}, \lambda_{\beta R}, \lambda_{dR}, \lambda_{cR}, \mu$.

However, for the first step of integration $t = t_1$, the optimal control of the defender is undefined. Hence, a different equation for $\dot{\mu}$ is used. Since at the time, when it enters the constrained segment $\lambda_{\beta R} = 0, \lambda_{dR} = 0$, the $h'(\mathbf{x})$ changes as follows:

$$h'(\mathbf{x}) = \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left[-v_D sin\hat{\theta}\dot{\hat{\theta}} + v_A sin\hat{\psi}\dot{\hat{\psi}}\right] \tag{3.196}$$

which reduces to

$$\frac{k_1}{k_2} = \left.\frac{\dot{k}_1}{\dot{k}_2}\right|_{t=t_1}$$

Substituting and solving for $\dot{\mu}$

$$sin\beta\left(-\lambda_{yAR}sin\beta\dot{\beta} - \frac{\dot{\lambda}_{\beta R}}{d} - \frac{\lambda_{\beta R}\dot{d}}{d^2}\right) = cos\beta\left(\lambda_{yAR}cos\beta\dot{\beta} - \dot{\lambda}_{dR} - \dot{\mu}\left(\frac{c_1+c_2}{v_A-v_D}\right)\right)$$

$$\dot{\mu}\left(\frac{c_1+c_2}{v_A-v_D}\right)cos\beta = \left(\lambda_{yAR}\dot{\beta} + \frac{\dot{\lambda}_{\beta R}sin(\beta)}{d} - \dot{\lambda}_{dR}cos\beta\right)$$

Rearranging for $\dot{\mu}$ the final equation is obtained as:

$$\dot{\mu} = \frac{\left(\lambda_{yA}\dot{\beta} + \frac{\dot{\lambda}_{\beta}sin\beta}{d} - \dot{\lambda}_{d}cos\beta\right)(v_A-v_D)}{(c_1+c_2)cos\beta} \tag{3.197}$$

After the initial step is taken, the equation (3.195) terms are calculated as follows:

$$\begin{aligned}
\frac{\partial k_1}{\partial x_A} &= 0 & \frac{\partial k_1}{\partial y_A} &= 0 \\
\frac{\partial k_1}{\partial \beta} &= -\lambda_{xA}v_A sin(\beta) + \lambda_{yA}v_A cos(\beta) & \frac{\partial k_1}{\partial d} &= 0 \\
\frac{\partial k_1}{\partial \lambda_{x_A}} &= v_A cos(\beta) & \frac{\partial k_1}{\partial \lambda_{y_A}} &= v_A sin(\beta) \\
\frac{\partial k_1}{\partial \lambda_{\beta}} &= 0 & \frac{\partial k_1}{\partial \lambda_d} &= -v_A \\
\frac{\partial k_1}{\partial \mu} &= \frac{c_1+c_2}{v_A-v_D}(-v_A) & \frac{\partial k_1}{\partial \lambda_c} &= 0 \\
\frac{\partial k_1}{\partial c} &= 0
\end{aligned} \tag{3.198}$$

$$\frac{\partial k_2}{\partial x_A} = 0 \qquad \frac{\partial k_2}{\partial y_A} = 0$$
$$\frac{\partial k_2}{\partial \beta} = -\lambda_{xA} v_A cos(\beta) - \lambda_{yA} v_A sin(\beta) \qquad \frac{\partial k_2}{\partial d} = \frac{\lambda_\beta v_A}{d^2}$$
$$\frac{\partial k_2}{\partial \lambda_{x_A}} = -v_A sin(\beta) \qquad \frac{\partial k_2}{\partial \lambda_{y_A}} = v_A cos(\beta)$$
$$\frac{\partial k_2}{\partial \lambda_\beta} = -\frac{v_A}{d} \qquad \frac{\partial k_2}{\partial \lambda_d} = 0$$
$$\frac{\partial k_2}{\partial \mu} = 0 \qquad \frac{\partial k_2}{\partial \lambda_c} = 0$$
$$\frac{\partial k_2}{\partial c} = 0 \tag{3.199}$$

$$\frac{\partial k_3}{\partial x_A} = 0 \qquad \frac{\partial k_3}{\partial y_A} = 0$$
$$\frac{\partial k_3}{\partial \beta} = 0 \qquad \frac{\partial k_3}{\partial d} = 0$$
$$\frac{\partial k_3}{\partial \lambda_{x_A}} = 0 \qquad \frac{\partial k_3}{\partial \lambda_{y_A}} = 0$$
$$\frac{\partial k_3}{\partial \lambda_\beta} = 0 \qquad \frac{\partial k_3}{\partial \lambda_d} = v_D$$
$$\frac{\partial k_3}{\partial \mu} = \frac{c_1 + c_2}{v_A - v_D}(v_D) \qquad \frac{\partial k_3}{\partial \lambda_c} = 0$$
$$\frac{\partial k_3}{\partial c} = 0 \tag{3.200}$$

$$
\begin{aligned}
\frac{\partial k_4}{\partial x_A} &= 0 & \frac{\partial k_4}{\partial y_A} &= 0 \\
\frac{\partial k_4}{\partial \beta} &= 0 & \frac{\partial k_4}{\partial d} &= \frac{-\lambda_\beta v_D}{d^2} \\
\frac{\partial k_4}{\partial \lambda_{x_A}} &= 0 & \frac{\partial k_4}{\partial \lambda_{y_A}} &= 0 \\
\frac{\partial k_4}{\partial \lambda_\beta} &= 0 & \frac{\partial k_4}{\partial \lambda_d} &= 0 \\
\frac{\partial k_4}{\partial \mu} &= 0 & \frac{\partial k_4}{\partial \lambda_c} &= 0 \\
\frac{\partial k_4}{\partial c} &= 0
\end{aligned} \tag{3.201}
$$

Using the above equations (3.198) to (3.201) along with the constrained control, adjoint equations, the value of μ is obtained along the constrained arcs for the set of all initial states $\mathbf{x}_0$. These help in characterizing the lower boundary of the retreat region $\mathbf{R}_R$ denoted by $m_2(\mathbf{x}) = 0$. This helps in generating a solution for the region $\mathbf{R}_4$.

An additional interior boundary constraint has to be taken into consideration when the value constraint becomes active. Let t_1 be the time when the value constraint becomes active. Then (t_1-) and (t_1+) represent the times just before and just after the constraint goes active. The state constraint (3.116) and control constraint (3.121) are rewritten as:

$$
g(\mathbf{x}) = c - a_1 + \left(\frac{c_1 + c_2}{v_A - v_D}\right)(d - d_c) \tag{3.202}
$$

$$
h(\mathbf{x}) = c_2 + \left(\frac{c_1 + c_2}{v_A - v_D}\right)\left(v_D cos\hat{\theta} - v_A cos\hat{\psi}\right) \tag{3.203}
$$

The state constraint creates interior boundary constraint on the adjoint variables which can be written as :

$$
\boldsymbol{\lambda}(t_1-) = \boldsymbol{\lambda}(t_1+) + \pi_1 \frac{\partial g}{\partial \mathbf{x}(t_1)} \tag{3.204}
$$

where g and h are taken from equations (3.202) and (3.203) and the $\boldsymbol{x}$ refers to the state variables $x_{AR}, y_{AR}, d_R, \beta_R, c$. So, equation(3.204) reduces to

$$\boldsymbol{\lambda}(t_1-) = \boldsymbol{\lambda}(t_1+) + \pi_1 \frac{\partial g}{\partial \mathbf{x}(t_1)} \tag{3.205}$$

Taking the partial derivative of equation(3.202) with respect to the state variables is:

$$\frac{\partial g}{\partial x_A} = 0 \tag{3.206}$$

$$\frac{\partial g}{\partial y_A} = 0 \tag{3.207}$$

$$\frac{\partial g}{\partial d} = \left(\frac{c_1 + c_2}{v_A - v_D}\right) \tag{3.208}$$

$$\frac{\partial g}{\partial \beta} = 0 \tag{3.209}$$

$$\frac{\partial g}{\partial c} = 1 \tag{3.210}$$

Using equation(3.206) in equation(3.205) it reduces to the following set of equations.

$$\lambda_{xAR}(t_1-) = \lambda_{xAR}(t_1+) \tag{3.211}$$

$$\lambda_{yAR}(t_1-) = \lambda_{yAR}(t_1+) \tag{3.212}$$

$$\lambda_{dR}(t_1-) = \lambda_{dR}(t_1+) + \pi_1 \left(\frac{c_1 + c_2}{v_A - v_D}\right) \tag{3.213}$$

$$\lambda_{\beta R}(t_1-) = \lambda_{\beta R}(t_1+) \tag{3.214}$$

$$\lambda_{cR}(t_1-) = \lambda_{cR}(t_1+) + \pi_1(1) \tag{3.215}$$

The equations (3.202) and (3.203) should be zero around the left and right limits of t_1. Thereby, the optimal control to satisfy this condition must be equal around the left and right limits of t_1. An important note is that at (t_1-) the state constraint is inactive. Thereby, the optimal control at t_1- would be

$$cos(\hat{\psi}^{R*})(t_1-) = \frac{k_1}{\rho_A} \tag{3.216}$$

$$cos(\hat{\theta}^{R*})(t_1-) = \frac{k_3}{\rho_D} \tag{3.217}$$

where

$$\begin{aligned} k_1 &= \lambda_{xAR}(t_1-)v_A cos\beta + \lambda_{yAR}(t_1-)v_A sin\beta \\ &\quad - \lambda_{dR}(t_1-)v_A \end{aligned} \tag{3.218}$$

$$\begin{aligned} k_2 &= -\lambda_{xAR}(t_1-)v_A sin\beta + \lambda_{yAR}(t_1-)v_A cos\beta \\ &\quad - \frac{\lambda_{\beta R}(t_1-)}{d} v_A \end{aligned} \tag{3.219}$$

$$k_3 = \lambda_{dR}(t_1-)v_D \tag{3.220}$$

$$k_4 = \frac{\lambda_{\beta R}(t_1-)v_D}{d} \tag{3.221}$$

$$\rho_A = \sqrt{k_1^2 + k_2^2} \tag{3.222}$$

$$\rho_D = \sqrt{k_3^2 + k_4^2} \tag{3.223}$$

The optimal control at t_1+ would be

$$cos(\hat{\psi}^{R*})(t_1+) = \frac{k_1}{\rho_A} \tag{3.224}$$

$$cos(\hat{\theta}^{R*})(t_1+) = \frac{k_3}{\rho_D} \tag{3.225}$$

where

$$k_1 = \lambda_{xAR}(t_1+)v_A cos\beta + \lambda_{yAR}(t_1+)v_A sin\beta$$
$$- \lambda_{dR}(t_1+)v_A - \mu(t_1+)\left(\frac{c_1+c_2}{v_A - v_D}\right)v_A \tag{3.226}$$
$$k_2 = -\lambda_{xAR}(t_1+)v_A sin\beta + \lambda_{yAR}(t_1+)v_A cos\beta$$
$$- \frac{\lambda_\beta(t_1+)}{d}v_A \tag{3.227}$$
$$k_3 = \lambda_{dR}(t_1+)v_D + \mu(t_1+)\left(\frac{c_1+c_2}{v_A - v_D}\right)v_D \tag{3.228}$$
$$k_4 = \frac{\lambda_{\beta R}(t_1+)v_D}{d} \tag{3.229}$$
$$\rho_A = \sqrt{k_1^2 + k_2^2} \tag{3.230}$$
$$\rho_D = \sqrt{k_3^2 + k_4^2} \tag{3.231}$$

Therefore,from equations(3.217) and (3.225):

$$\left.\frac{k_3}{\rho_D}\right|_{t=t_{1-}} = \left.\frac{k_3}{\rho_D}\right|_{t=t_{1+}} \tag{3.232}$$

Squaring (3.232) on both sides

$$1 + \frac{\left(\frac{\lambda_{\beta R}(t_1-)}{d}\right)^2}{\lambda_{dR}^2(t_1-)} = 1 + \frac{\left(\frac{\lambda_{\beta R}(t_1+)}{d}\right)^2}{\left(\lambda_{dR}(t_1+) + \mu(t_1+)\left(\frac{c_1+c_2}{v_A-v_D}\right)\right)^2}$$

Using equation (3.214) it reduces to:

$$\lambda_{dR}(t_1-) = \lambda_{dR}(t_1+) + \mu(t_1+)\left(\frac{c_1+c_2}{v_A - v_D}\right) \tag{3.233}$$

We have another equation relating the adjoint variables just before and after t_1 which is equation(3.213). Comparing this equation with equation (3.233) we get,

$$\pi_1 = \mu(t_1+) \tag{3.234}$$

Using the optimal control of the attacker a similar result is obtained. From equations (3.216) and (3.224) :

$$\left.\frac{k_1}{\rho_A}\right|_{t=t_1-} = \left.\frac{k_1}{\rho_A}\right|_{t=t_1+}$$

$$\begin{aligned}
d &= (\lambda_{xAR}(t_1-)v_A cos\beta + \lambda_{yAR}(t_1-)v_A sin\beta - \lambda_{dR}(t_1-)v_A) \\
e &= (\lambda_{xAR}(t_1-)v_A cos\beta + \lambda_{yAR}(t_1-)v_A sin\beta - \lambda_{dR}(t_1-)v_A)^2 \\
f &= \left(-\lambda_{xAR}(t_1-)v_A sin\beta + \lambda_{yAR}(t_1-)v_A cos\beta - \frac{\lambda_{\beta R}(t_1-)}{d}v_A\right)^2
\end{aligned} \tag{3.235}$$

$$\begin{aligned}
a &= \lambda_{xAR}(t_1+)v_A cos\beta + \lambda_{yAR}(t_1+)v_A sin\beta - \lambda_{dR}(t_1+)v_A - \mu(t_1+)\frac{(c_1+c_2)}{v_A - v_D}v_A \\
b &= \left(\lambda_{xAR}(t_1+)v_A cos\beta + \lambda_{yAR}(t_1+)v_A sin\beta - \lambda_{dR}(t_1+)v_A - \mu(t_1+)\frac{(c_1+c_2)}{v_A - v_D}v_A\right)^2 \\
c &= \left(-\lambda_{xAR}(t_1+)v_A sin\beta + \lambda_{yAR}(t_1+)v_A cos\beta - \frac{\lambda_{\beta R}(t_1+)}{d}v_A\right)^2
\end{aligned}$$

The L.H.S of the equation is:

$$\frac{d}{\sqrt{e+f}} \tag{3.236}$$

The R.H.S is:

$$\frac{a}{\sqrt{b+c}} \tag{3.237}$$

From (3.236) and (3.237),(3.214),(3.211),(3.212) we get

$$-\lambda_{dR}(t_1-) = -\lambda_{dR}(t_1+) - \mu(t_1+)\left(\frac{c_1+c_2}{v_A - v_D}\right) \tag{3.238}$$

We have another equation relating the adjoint variables just before and after t_1 which is equation(3.213).Comparing this equation with equation (3.238) we get

$$\pi_1 = \mu(t_1+) \tag{3.239}$$

A dispersal surface exists where the global equilibrium solution divides the admissible state space into two regions that contain qualitatively different equilibrium control strategies. The mobile attacker could retreat from any of the two optimal paths provided. So, the symmetry of all such initial states constitutes a dispersal surface.

Theorem 5: The surface $\mathbf{S} := \{\mathbf{x} \in \mathbf{R}_A | \beta = -\frac{\pi}{2}\}$ *represents a dispersal surface within the global equilibrium solution of the game.*

Proof: A symmetric solution $\bar{\mathbf{x}}(t;\mathbf{x_0}), \bar{\lambda}(t;\mathbf{x_0}), \bar{\bar{\psi}}(t;\mathbf{x_0}), \bar{\bar{\theta}}(t;\mathbf{x_0})$ can be constructed by switching the sign of $\lambda_\beta(t;\mathbf{x_0})$ and integrating forward in time. Let $\bar{\mathbf{x}}_0 = \mathbf{x}_0$ and $\bar{\lambda}_\mathbf{0} = (\bar{\lambda}_{xA}, \bar{\lambda}_{yA}, -\bar{\lambda}_\beta, \bar{\lambda}_d, \bar{\lambda}_c)$.

From the equilibrium feedback strategies, we observe that a change in sign of $\bar{\lambda}_\beta(t;\mathbf{x_0})$ implies,

$$\bar{\lambda}_{xAR}(t;\mathbf{x_0}) = \lambda_{xAR}(t;\mathbf{x_0}) \tag{3.240}$$

At the dispersal surface $\beta = -\frac{\pi}{2}$,

$$\begin{aligned}
\bar{k}_2 &= -\bar{\lambda}_{xAR} v_A sin\beta + \bar{\lambda}_{yAR} v_A cos\beta - \frac{\bar{\lambda}_{\beta R}}{d} v_A \\
&= -\frac{\bar{\lambda}_{\beta R}}{d} v_A \\
&= \frac{\lambda_{\beta R}}{d} v_A \\
&= -k_2 \qquad (3.241) \\
\bar{k}_4 &= \frac{\bar{\lambda}_{\beta R} d}{v_D} \\
&= \frac{\lambda_{\beta R} d}{v_D} \\
&= -k_4 \qquad (3.242)
\end{aligned}$$

The optimal control for the attacker is therefore:

$$\begin{aligned}
tan(\bar{\hat{\psi}}^{R*}) &= \frac{\bar{k}_2}{k_1} \\
&= -\frac{k_2}{k_1} \\
&= -tan(\hat{\psi}^{R*}) \\
&= tan(-\hat{\psi}^{R*}) \\
\bar{\hat{\psi}}^{R*} &= -\hat{\psi}^{R*} \qquad (3.243)
\end{aligned}$$

The optimal control for the defender is therefore:

$$\begin{aligned}
tan(\bar{\hat{\theta}}^{R*}) &= \frac{\bar{k}_4}{k_3} \\
&= -\frac{k_4}{k_3} \\
&= -tan(\hat{\theta}^{R*}) \\
&= tan(-\hat{\theta}) \\
\bar{\hat{\theta}}^{R*} &= -\hat{\theta}^{R*}
\end{aligned} \tag{3.244}$$

Substituting these initial conditions in the state dynamics,the following is observed.

$$\begin{aligned}
\bar{\dot{d}} &= -v_A cos\bar{\hat{\psi}} + v_D cos\bar{\hat{\theta}} \\
&= v_D cos(-\hat{\theta}) - v_A cos(-\hat{\psi}) \\
&= v_D cos(\hat{\theta}) - v_A cos(\hat{\psi}) \\
&= \dot{d}
\end{aligned} \tag{3.245}$$

Thereby, integrating forward in time:

$$\bar{d} = d \tag{3.246}$$

From the tangency line, we know that

$$\bar{\beta} = \pi - \beta \tag{3.247}$$

Similarly,

$$\begin{aligned}
\dot{\bar{x}}_A &= v_A cos(\bar{\hat{\psi}} + \beta) \\
&= v_A cos((-\hat{\psi}) + \pi - \beta) \\
&= v_A cos(\pi - (\hat{\psi} + \beta)) \\
&= -v_A cos(\hat{\psi} + \beta) \\
&= -\dot{x}_A
\end{aligned} \tag{3.248}$$

and

$$\begin{aligned}
\dot{\bar{y}}_A &= v_A sin(\bar{\hat{\psi}} + \beta) \\
&= v_A sin((-\hat{\psi}) + \pi - \beta) \\
&= v_A sin(\pi - (\hat{\psi} + \beta)) \\
&= v_A sin(\hat{\psi} + \beta) \\
&= \dot{y}_A
\end{aligned} \tag{3.249}$$

From equations (3.246) and (3.247), it can be seen that the d and y_A components are equal and both solutions terminate at the same terminal time: $t_f = \bar{t}_f$. The utilities of the attacker and defender are equal, the states contained in $\mathbf{S}$ satisfy the condition for a dispersal surface.

3.3 Overall Engage or Retreat Game

There may exist values for the initial state $\mathbf{x}_0$ in which the control strategies $u_A(t)$ and $\mathbf{u}_D(t)$ do not exist, that satisfy the constraint and boundary conditions. This causes a divide in the solution dividing the admissible state space $\mathbf{R}_A$ into two disjoint regions. A region which has states where the solution to the OCR exists and a second region which has states where a solution does not exist.

These are defined as:

$$\mathbf{R}_R := \mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_4 \tag{3.250}$$

$$\mathbf{R}_E := \mathbf{R}_3 \tag{3.251}$$

The overall solution to the ERG is found by identifying which region the state $\mathbf{x}_0$ belongs to and by implementing the control strategies corresponding to that region as defined in the following theorem.

Theorem 6: Suppose that regions $\mathbf{R}_R$ and $\mathbf{R}_E$ can be known or calculated. Along the boundary of $\mathbf{R}_R$ suppose that $V_R(\mathbf{x}) \geq V_E(\mathbf{x})$. The following control strategies constitute a Nash Equilibrium for the Engage or Retreat Game defined in (3.250) and (3.251).

$$u_A^*(t;\mathbf{x}_0) = \begin{cases} u_A^{*E}(t;\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \\ u_A^{*R}(t;\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \end{cases}$$

$$\mathbf{u}_D^*(t;\mathbf{x}_0) = \begin{cases} \mathbf{u}_D^{*E}(t;\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \\ \mathbf{u}_D^{*R}(t;\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \end{cases}$$

The resulting Nash Equilibrium utilities for each player are

$$U_A^*(\mathbf{x}_0) = \begin{cases} V_E(\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_E \\ V_R(\mathbf{x}_0) & \mathbf{x}_0 \in \mathbf{R}_R \end{cases}$$

and

$$U_B^*(\mathbf{x}_0) = \begin{cases} b_1 & \mathbf{x}_0 \in \mathbf{R}_E \\ 0 & \mathbf{x}_0 \in \mathbf{R}_R \end{cases}$$

The equilibrium strategies, state trajectories and adjoint values are given by the solutions of the GoE or OCR that correspond to the equilibrium control strategies. The proof for this theorem can be found in [1].

Numerical Solutions

In this chapter, the equilibrium trajectories and regions will be examined for the two related optimization problems namely the Game of Engagement and the Optimal Constrained Retreat. Numerical solutions are obtained for the differential game. The following parameter values are used for the GoE.

$v_A = 1.5 \quad v_D = 1 \quad c_1 = 1 \quad c_2 = 1 \quad d_c = 1 \quad y_R = -4$. For, the OCR and ERG the attacker's velocity is increased to $v_A = 3.5$, so that the results are visually clear and are easier to explain.

4.1 Game of Engagement

The Figure 4.1 shows a numerical example for the GoE in the global coordinate system. The capture surface is depicted as a unit circle $\sqrt{x^2 + y^2} = 1$. The trajectories of the mobile attacker and mobile defender are in red and blue respectively as seen in Figure 4.1. The initial position of the attacker and defender are $\mathbf{x_0} = (x_{A0}, y_{A0}, x_{D0}, y_{D0}) = (0, 0, -5, -5)$. The attacker attacks the defender and captures it when the distance between them reduces to the capture radius and is represented by a red cross on the capture circle. Figure 4.2

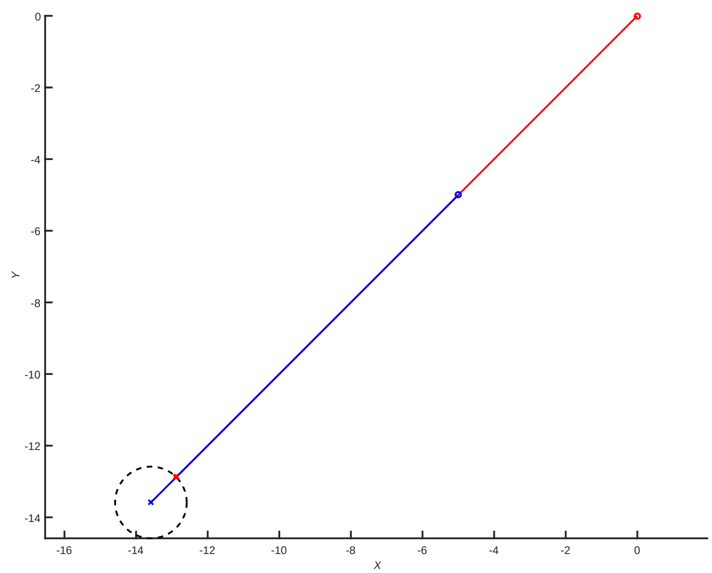

Figure 4.1: Global Coordinate Solution-Initial state 1

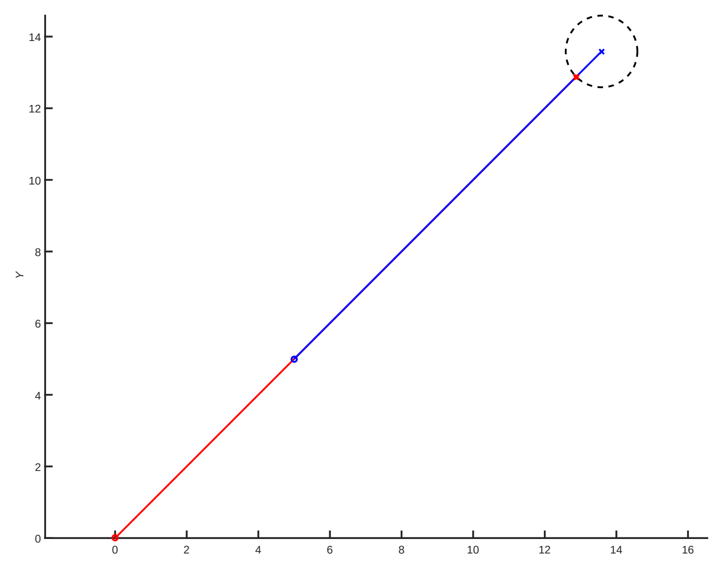

Figure 4.2: Global Coordinate Solution-Initial state 2

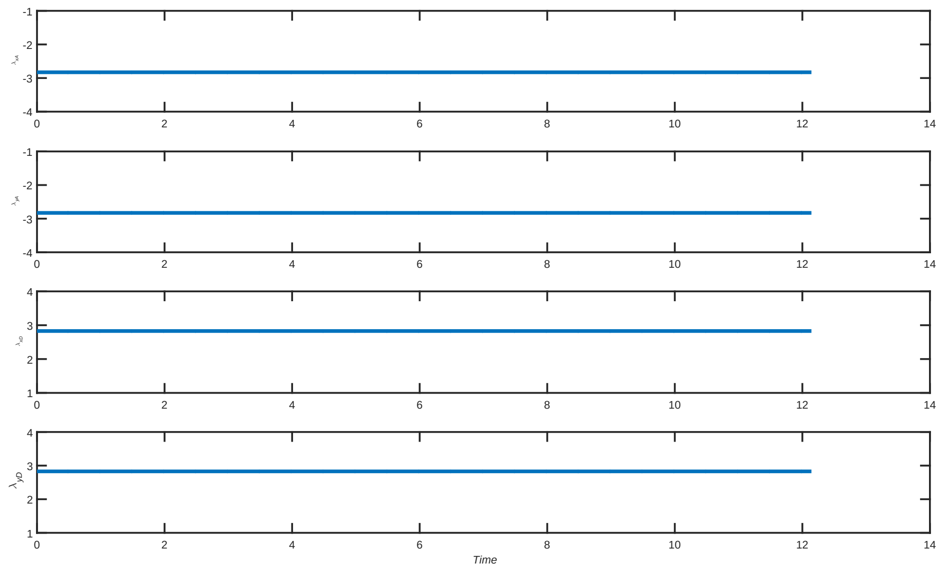

Figure 4.3: Game Of Engagement Adjoint Variables

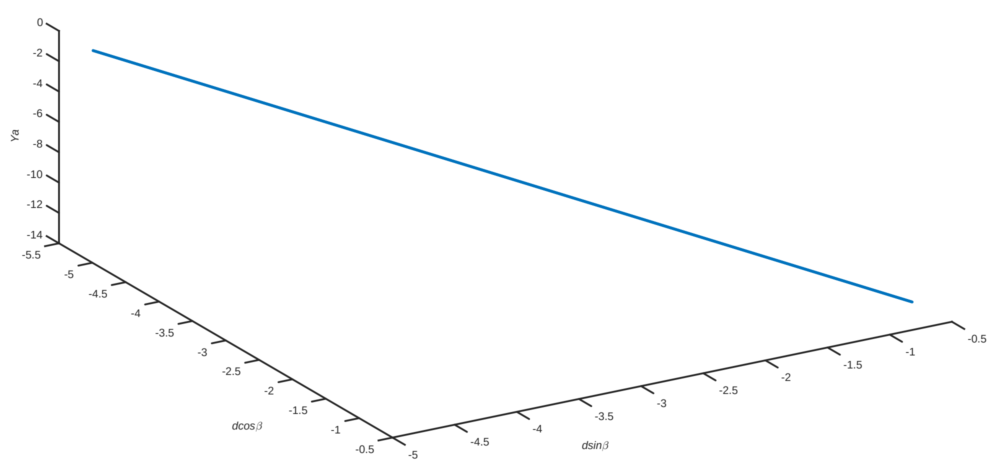

Figure 4.4: Relative Solution

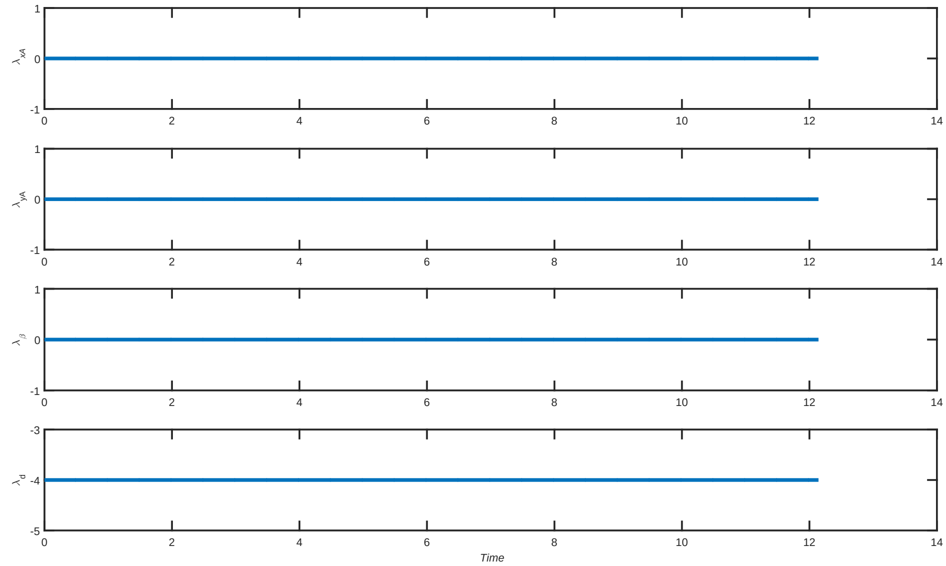

Figure 4.5: Relative Solution Adjoint Variables

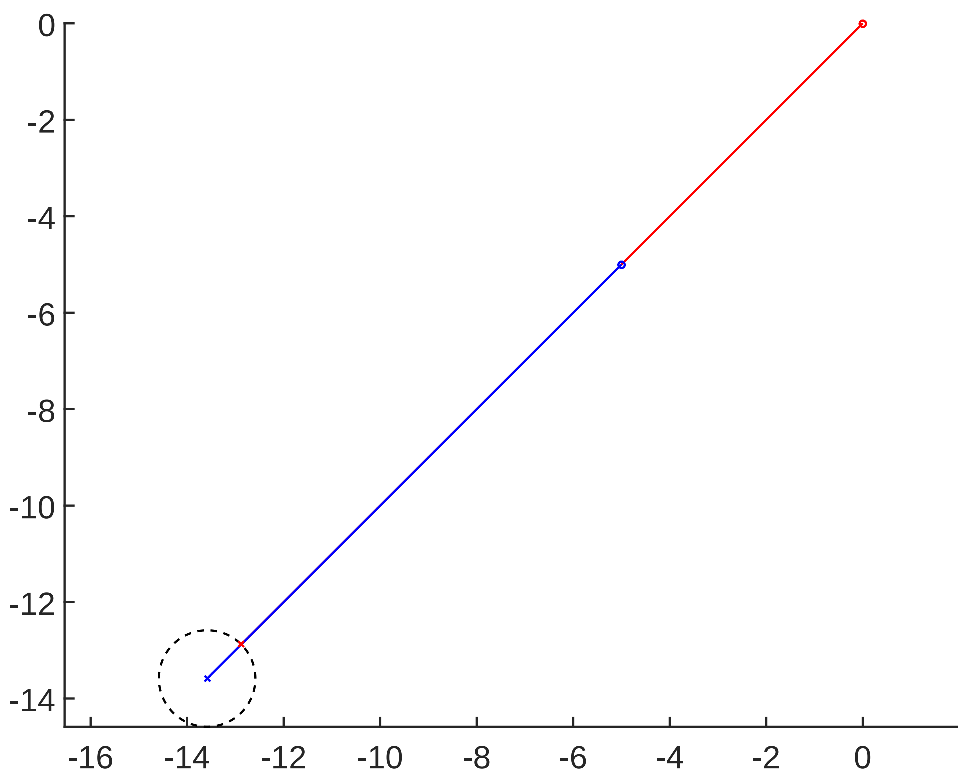

Figure 4.6: Relative to Global Converted Solution

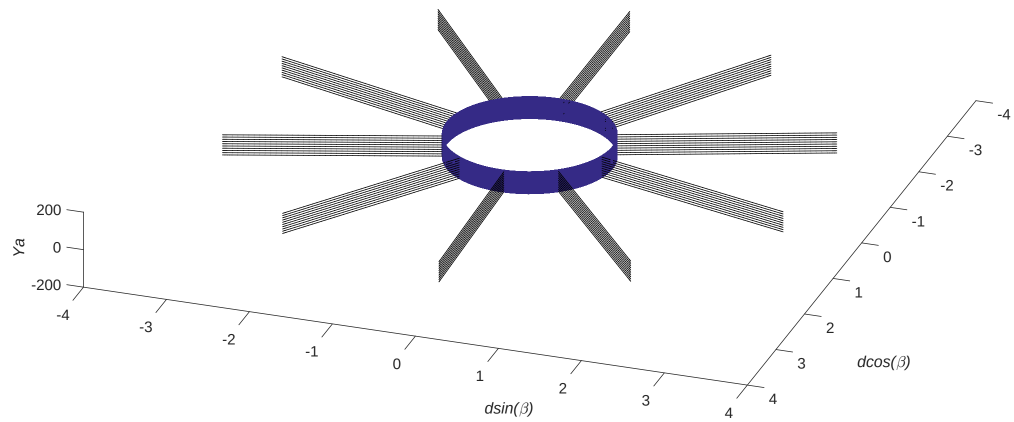

Figure 4.7: Game Of Engagement-Relative Multiple

shows the trajectories for the initial state of $\mathbf{x_0} = (x_{A0}, y_{A0}, x_{D0}, y_{D0}) = (0, 0, 5, 5)$. The adjoint variables of the mobile attacker and mobile defender throughout the game of attack are as shown in Figure 4.3. These were obtained for the equilibrum trajectories shown in Figure 4.1. The adjoint variables of the attacker and the defender have the same magnitude but have their signs reversed. The adjoint variables for the mobile attacker and the mobile defender are constant throughout the game of attack.

The initial condition to generate the numerical solution in the Relative system is $\mathbf{x_0} = (x_{A0}, y_{A0}, d_0, \beta_0) = (0, 0, -2.325, 7.0711)$. A sample optimal trajectory is seen in Figure 4.4. To plot the trajectory for the game of attack in the Relative coordinate systems the following axes are used. The x-axis is $dcos(\beta)$. The y-axis is $dsin(\beta)$. The z-axis is the y co-ordinate of the attacker y_A. The equilibrium trajectory shows that for the mobile attacker to capture the mobile defender, the distance d is the only state component that changes. This is further illustrated by the Figure 4.5 where except the adjoint variable λ_d, the other adjoint variables are zero. To show that the Global and Relative coordinate systems are equivalent to each other, the same initial condition used to generate a numerical solution in Global system is converted to the Relative System using equations (2.14) to (2.17). The Figure 4.6

shows the result obtained in relative coordinate system when converted to global coordinate system. The trajectories of the mobile attacker and the mobile defender are the same as in Figure 4.1.If the equilibrium trajectories are plotted for a range of initial conditions $\mathbf{x}_0$ the Figure 4.7 is obtained. The cylinder represents the capture surface. The radius is the capture distance and the height is the attacker's y co-ordinates. The mobile attacker can be seen to capture the mobile defender for all $\mathbf{x}_0$ in $\mathbf{R}_3$.

4.2 Optimal Constrained Retreat

In the figures from 4.8 to 4.13 the capture surface is depicted as a cylinder along the Z-axis. The retreat surface is shown as a mesh plane at $y_r = -4$. The equilibrium trajectories for regions for $\mathbf{R}_1 \cup \mathbf{R}_2$ are shown in Figures 4.8 and 4.9. The blue trajectories show the solution for several initial states. The upper bound is $m_1(\mathbf{x})$ for the unconstrained trajectories of $\mathbf{R}_1 \cup \mathbf{R}_2$. The tangency lines are depicted by red lines $(d * cos\beta_{T1}, d * sin\beta_{T1}, y_{AT})$ and $(d * cos\beta_{T2}, d * sin\beta_{T2}, y_{AT})$.

In Figure 4.10, the blue trajectories reach the lower bound $m_2(\mathbf{x})$ tangentially and travel along the lower bound till they approach the tangency lines depicted in red. The surface $\beta = \frac{-\pi}{2}$ is the dispersal surface. Two trajectories on either side of the tangency lines can be seen traveling to reach the retreat surface in Figure 4.11 in region $R4$. The escort region is represented by the region $\mathbf{R}_4$. In this region, the defender cooperates with the attacker to maximize the utility of the attacker. The attacker follows a constrained retreat trajectory so that it does not enter the region of engagement. The defender escorts the attacker till the state reaches the tangency line away from where engagement could be optimal.

Figure 4.13 displays the equilibrium trajectories in black for the Game of Engagement where the attacker engages the defender and terminates the game at the capture surface. This is the region of engagement $\mathbf{R}_3$ and it is enclosed by the upper boundary $m_1(\mathbf{x}) = 0$

and by the lower boundary $m_2(\mathbf{x}) = 0$. Figure 4.12 shows sample equilibrium trajectories for both tangency lines obtained for regions $\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4$ converted to the global coordinate system. The black and red lines represent the trajectories of the mobile attacker and mobile defender from tangency line 1. The tangency line is represented by a 'x' marker. The attacker travels and enters the constrained region tangentially at t_1 and leaves the arc at t_2 to travel and reach the retreat surface. The defender can be seen to travel away from the attacker. The defender effectively avoids capture by the attacker as can be seen from the trajectories.

In the following Figures, 4.14,4.15,4.16 the adjoint variables are shown. The red lines indicate the unconstrained segments and the blue line indicates the constrained segment. A discontinuity can be seen in the $\lambda_d(t)$ and $\mu(t)$ at $t_1 = 81.98s$. This is due to the appearance of the interior boundary condition when the constraint becomes active.

In the Figures 4.17,4.18 the control of the mobile attacker and mobile defender are shown. The control is continuous. The mobile attacker's control is seen to be constant during the time interval (t_1, t_2). This is the time when the constraint is active. The mobile defender's control varies during the time interval implying that it cooperates with the attacker and escorts it around regions where engagement could occur.

In Figure 4.19, the dashed black line represents the value function for the Game of Engagement, along $V_E(\mathbf{x}_R^*(t))$. The trajectory can be seen entering the constraint tangentially at t_1 and leaving it tangentially at t_2.

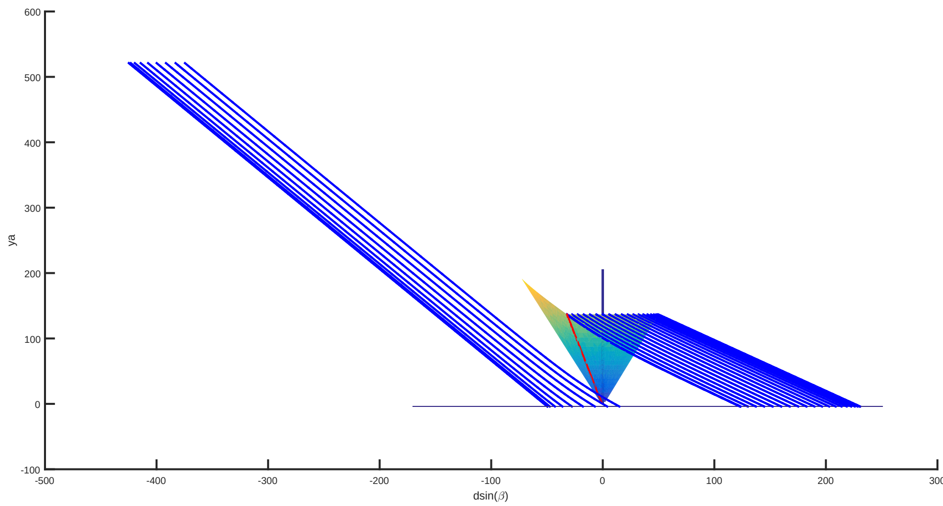

Figure 4.8: Equilibrium Trajectories for $\mathbf{R}_1 \cup \mathbf{R}_2$

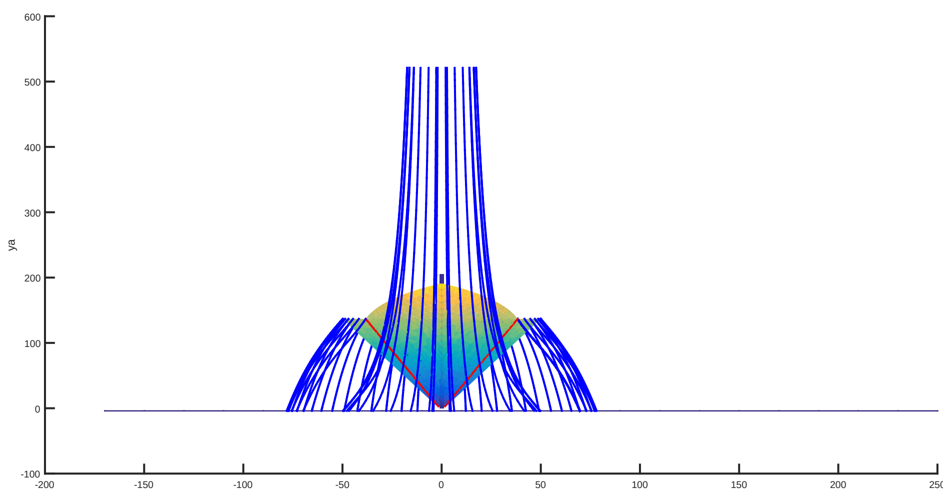

Figure 4.9: Equilibrium Trajectories for $\mathbf{R}_1 \cup \mathbf{R}_2$

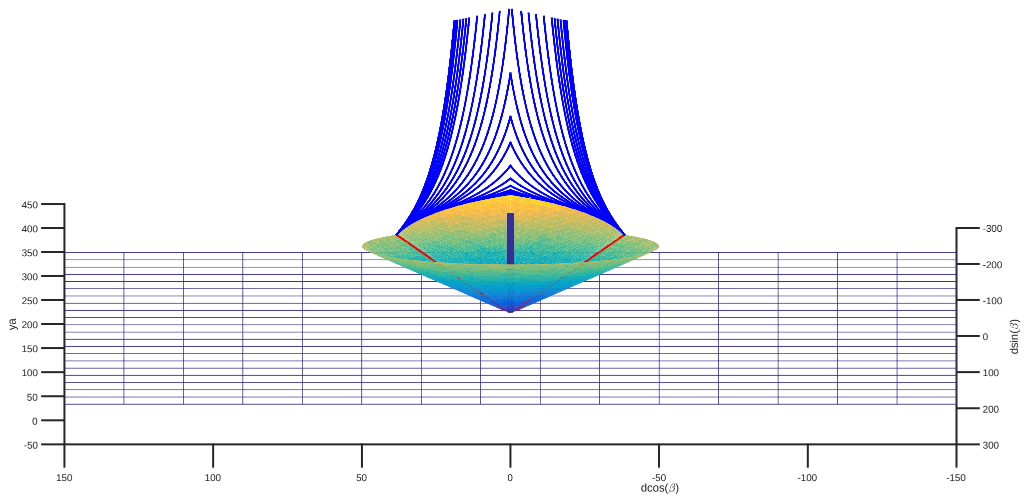

Figure 4.10: Equilibrium Trajectories for $\mathbf{R}_4$

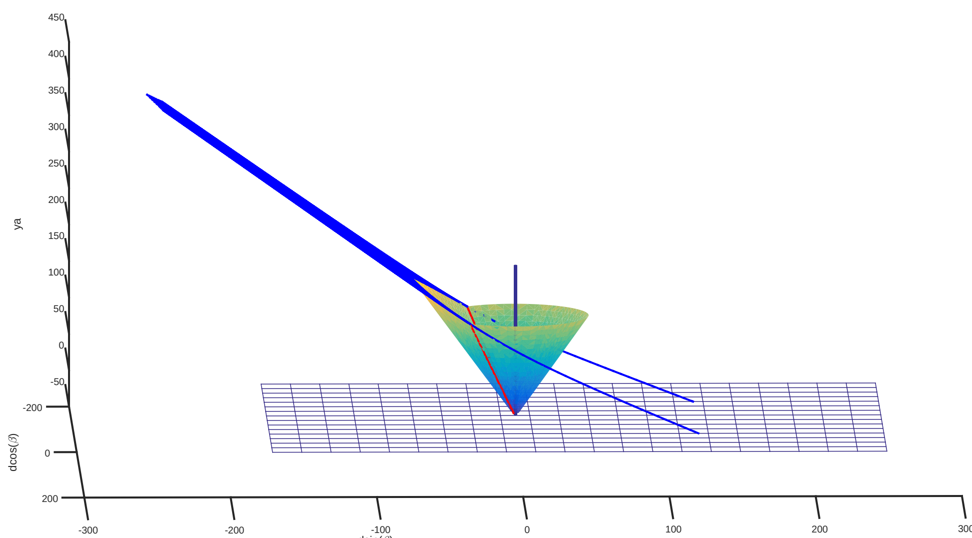

Figure 4.11: Equilibrium Trajectories for $\mathbf{R}_4$

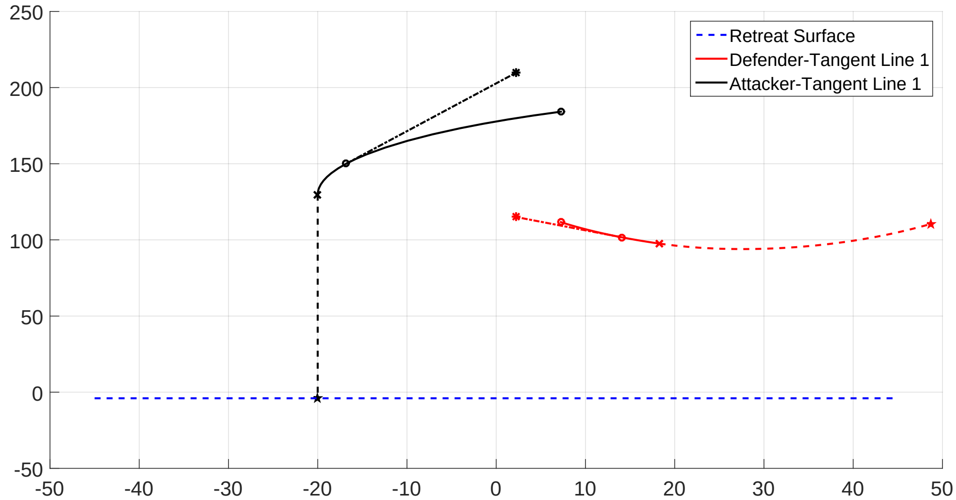

Figure 4.12: OCR solution R2,R3,R4 in Global coordinates

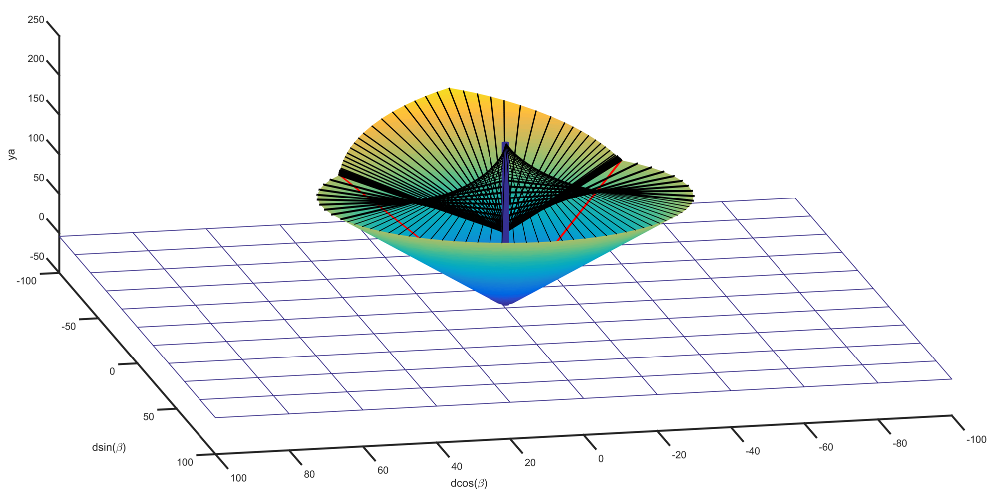

Figure 4.13: Equilibrium Trajectories for $\mathbf{R}_3$

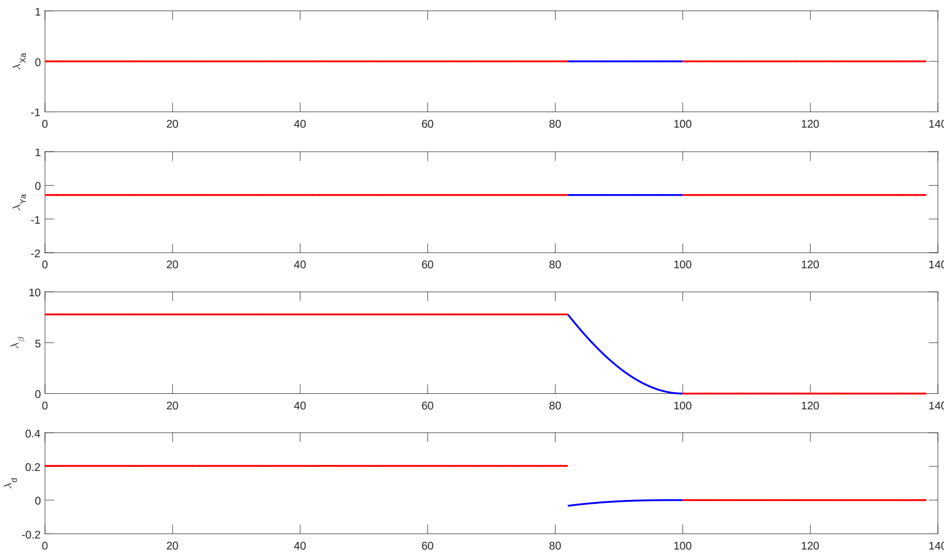

Figure 4.14: Adjoint Variables for Tangency Line 1

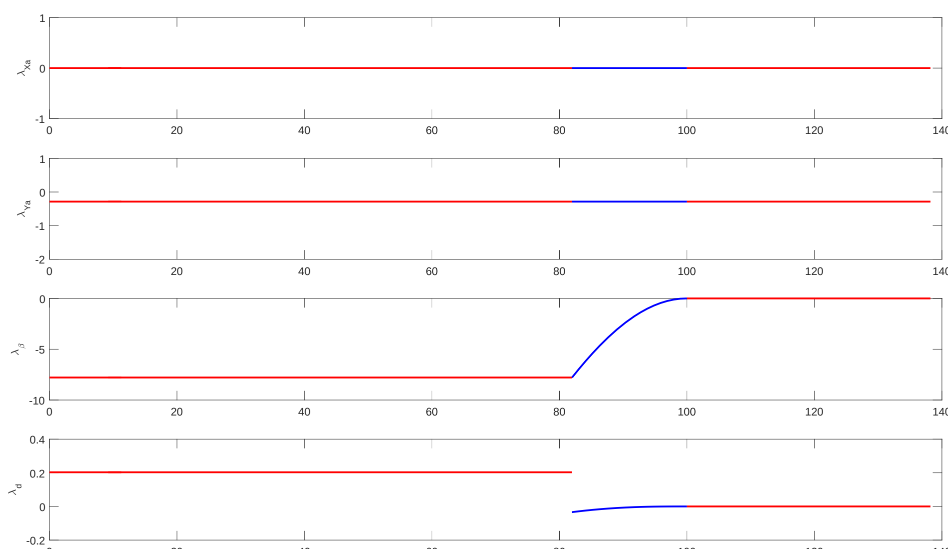

Figure 4.15: Adjoint Variables for Tangency Line 2

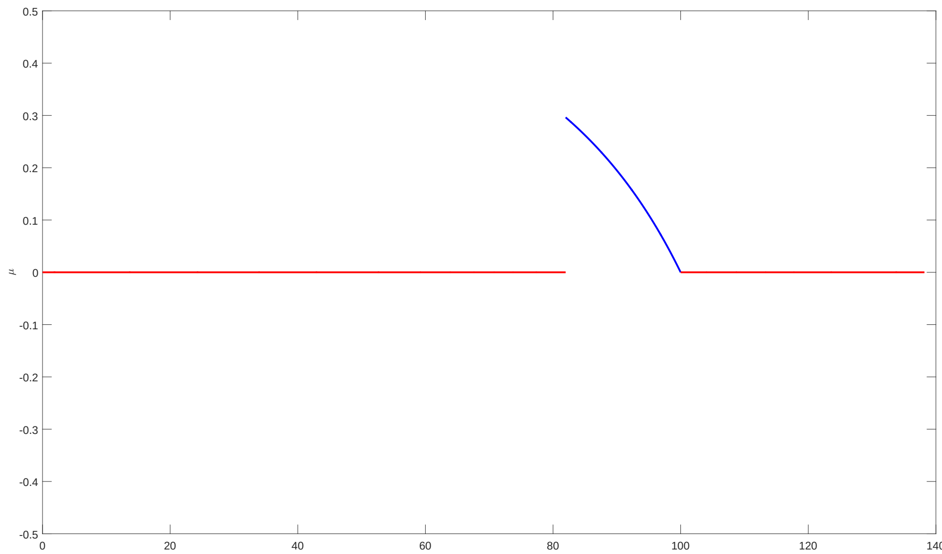

Figure 4.16: Control Constraint

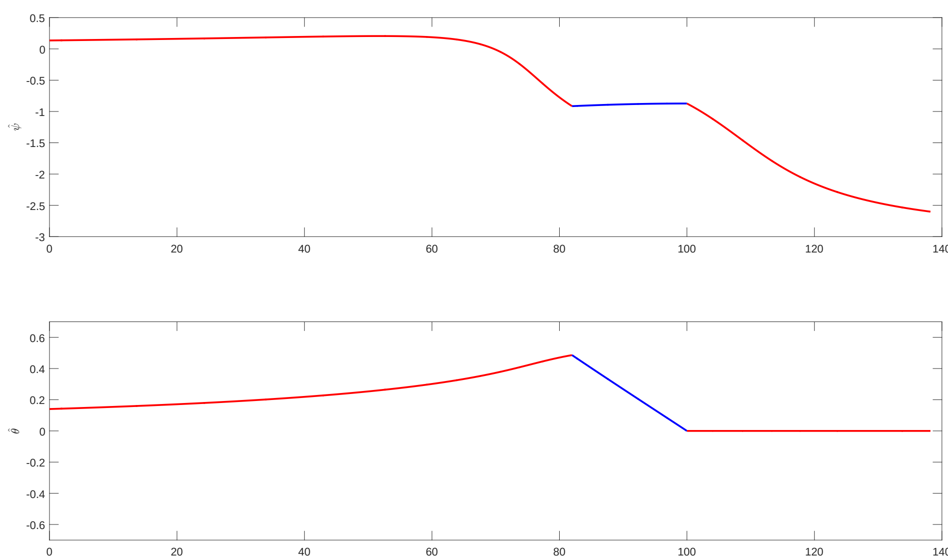

Figure 4.17: Optimal Control for Tangency Line 1

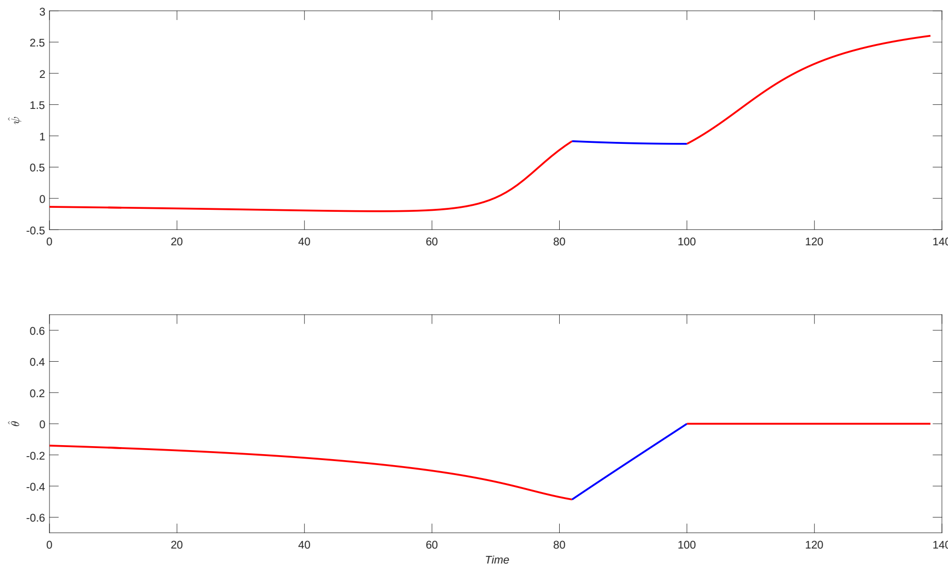

Figure 4.18: Optimal Control for Tangency Line 2

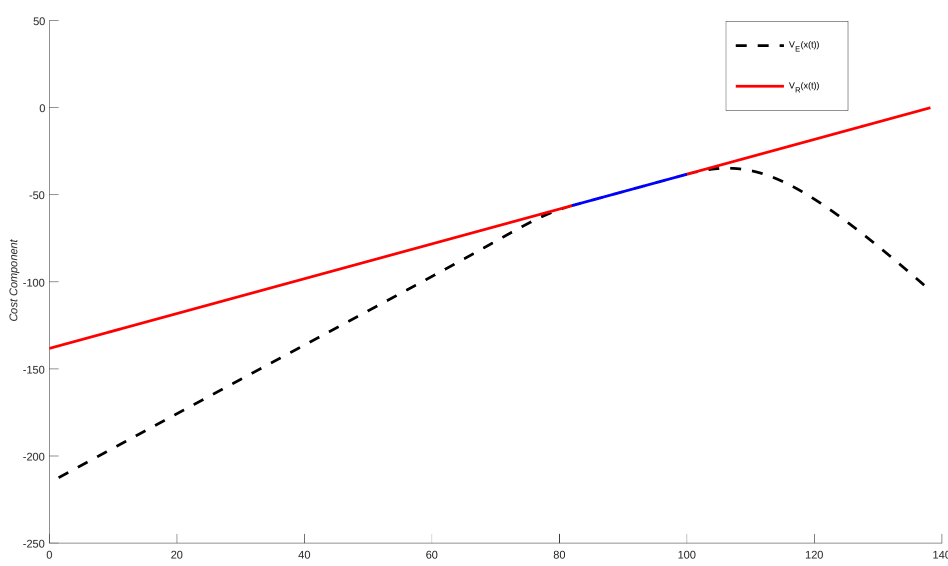

Figure 4.19: Cost Component

4.3 Overall Engage or Retreat Game

Figure 4.20 shows the complete solution to the Engage or Retreat differential game by combining the equilibrium trajectories for all admissible states which are represented by $\mathbf{R}_A$ which consists of all the states $\mathbf{x}_0$ for which solutions exist.

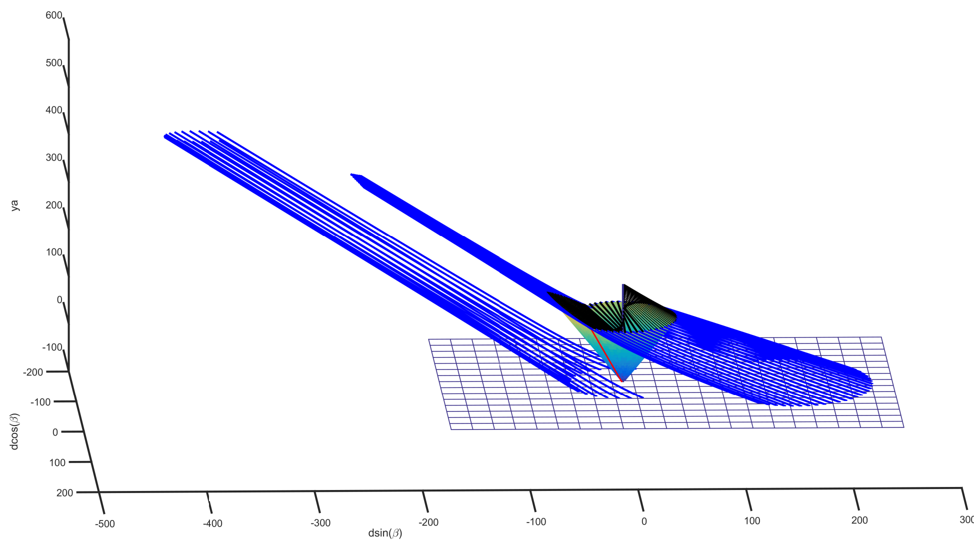

Figure 4.20: Equilibrium Trajectories for $\mathbf{R}_A$

Conclusion

The thesis solves an Engage or Retreat differential Game for mobile agents. Solutions for two related optimization problems are obtained and proven as equilibrium solutions to the ERG. For certain initial states, it is optimal that the defender should cooperate with the attacker to maximize its utility function so that retreat looks like the best option available. This is achieved by imposing a constraint on the Value function for the OCR which produces of constrained retreat called *escort regions*. This is the region where the defender and attacker work together to avoid entering a region where engagement becomes optimal. The Value functions are equal during the constrained portions. Although a discontinuity, is seen in the adjoint variables $\lambda_d(t), \mu(t)$, the control of the agents is seen to be continuous. This game structure could be extended to include multiple defender systems.

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